§1. Introduction. In treatises or advanced textbooks on theoretical physics, it is apparent that the way mathematics is used is very different from what is to be found in books of mathematics. There is, for example, no close connection between books on analysis, on the one hand, and any classical textbook in quantum mechanics, for example, Schiff, [11], or quite recent books, for example Ryder, [10], on quantum field theory. The differences run a good deal deeper than the fact that the books on theoretical physics are not written in the definition-theorem-proof style characteristic of pure mathematics. Although a good many propositions are proved in the books on physics, there are almost with exception no existential proofs, and consequently there is no really serious systematic use of quantifiers. Another important characteristic is the free use of infinitesimals. In fact, most results would not lose anything, from a physicist’s point of view, by leaving them in approximate form, i.e., instead of strict equalities or inequalities, using equalities or inequalities only up to an infinitesimal.

The discrepancy between the way mathematics is ordinarily done in theoretical physics and the way it is built up from a foundational standpoint in any of the standard modern views raises the question of whether it might be possible to construct quite directly a rigorous foundation that reflects a significant part of this standard practice in theoretical physics. Other parts of standard practice in physics, for example, the use of physically intuitive but nonrigorous arguments, are not present in our system.

To reflect the features mentioned above that are characteristic of works in theoretical physics, the foundational approach we develop here has the following properties:

(i) The formulation of the axioms is essentially a free-variable one with no use of quantifiers.

(ii) We use infinitesimals in an elementary way drawn from nonstandard analysis, but the account here is axiomatically self-contained and deliberately elementary in spirit.
(iii) Theorems are left only in approximate form; that is, strict equalities and inequalities are replaced by approximate equalities and inequalities. In particular, we use neither the notion of standard function nor the standard part function. Such approximations are not explicit in physics, but can be viewed as implicit in the way infinitesimals are used.

By limiting ourselves to a fragment with these three features, we are able to prove the consistency of our theory by finitary means (§2). The proof first reduces the system to a fragment of the theory of rational numbers with infinitesimals, and then uses Herbrand's theorem for this fragment. The fact that the theory can be reduced to rational numbers is typical of nonstandard analysis, where many proofs and computations have an algebraic character. It is also to be noted that the axioms are true in models for nonstandard analysis.

The kind of system we propose satisfies at least in part a central goal of Hilbert's program of proof theory, namely to give an axiomatic foundation of analysis sufficient for the expression of geometry and physical theories but which can at the same time be proved consistent. This concern for including the foundations of physics runs throughout Hilbert's career and is expressed in one of his last publications on the foundations of mathematics ([4], p. 472 of the English translation). We certainly have not covered as much ground as possible, for extension to functions of several variables is needed, but the direction we have taken is a constructive one that should be able to encompass a large portion of current theoretical physics and be at the same time finitarily consistent. On the other hand, our system is not nearly as finitistic as Mycielski's system [6]. More generally, we think that finitistic constructive systems cannot directly reflect the way mathematics is done in much of theoretical physics, as our system can.

We give a brief summary of the contents of the paper. In §1 we present the axioms and in §2 the finitary consistency proof, using Herbrand's theorem. The rest of the paper gives a flavor of what can be done in the system: §§3 and 4 on differentials, derivatives and continuous functions, §5 on the nonstandard principles of overflow, §§6 and 7 on integrals, and §8 on series and transcendental functions. It may be added that differential equations may be treated in the present system without major difficulties. For examples on differential equations and more detailed proofs, see [12].

§2. Axioms. We assume full first-order logic with equality, where the variables range over numbers, but the axioms are almost free-variable in form. Constants for numbers include 0, 1, and \( v_0 \), which stands for an infinite natural number. Constants for functions include \( +, \cdot, / \) (division), \( I \) (the identity function introduced by \( I(x) = x \)), and \( C_\tau \), where \( \tau \) is any constant term, introduced by \( C_\tau(x) = \tau \). We have the binary predicate \(<\), and the unary predicates \( \text{Inf} \) and \( \mathcal{N} \). Later on we shall introduce expressions for derivatives and integrals. The formula \( \text{Inf}(x) \) is interpreted as "\( x \) is infinitesimal" and \( \mathcal{N}(x) \) as "\( x \) is a natural number (finite or infinite)". Internal formulas are those where neither \( \text{Inf} \) nor expressions defined from \( \text{Inf} \) occur. Other formulas are external.

We have the full set of open axioms for an ordered field, including \( 0 \neq 1 \). Instead of introducing the absolute value, we add a function \( \delta \) with the following axiom.
Axiom 1. (1) \( x \geq 0 \rightarrow \delta(x) = 1 \).
(2) \( x < 0 \rightarrow \delta(x) = -1 \).

With this function, we can define \( |x| = \delta(x) \cdot x \). The function \( \delta \) is internal.

We also have the axioms for Peano arithmetic, but with a schema for a minimum operator restricted to open internal formulas replacing a schema of open internal induction. It is well known that the two schemas are equivalent. The minimum schema is the following. We adopt the convention that the variables are in a list and that \( k \) is the first variable in that list. When we write \( \varphi(x) \) we mean that \( x \) is substituted for the first variable. Since we only use open formulas, \( k \) is always free for \( x \).

Axiom 2. Let \( \varphi \) be an internal open formula, where neither \( N \) nor \( \min \) occur and \( x_1, \ldots, x_n \) are the distinct free variables in \( \varphi \), except for the first variable. We introduce an \( n \)-ary function symbol \( \min_\varphi \) with the following axiom:

\[
N(x) \land \varphi(x) \rightarrow N\left(\min_\varphi(x_1, \ldots, x_n)\right) \land \min_\varphi(x_1, \ldots, x_n) \leq x
\]

\[
\land \varphi\left(\min_\varphi(x_1, \ldots, x_n)\right).
\]

We may omit the variables \( x_1, \ldots, x_n \), when they are clear from the context. This axiom implies Herbrand's minimum axiom as presented in [3]. In order to imitate Herbrand's case, for each formula \( \varphi \), with the same restrictions as for our axiom, we must introduce an \((n + 1)\)-ary function symbol \( \epsilon_\varphi \) such that (as usual, we omit the extra variables, and use use \( n \) and \( m \) as variables for natural numbers)

(1) \( \epsilon_\varphi(0) = 0 \),
(2) \( \neg \varphi(0) \land \varphi(n + 1) \land \epsilon_\varphi(n) = 0 \rightarrow \epsilon_\varphi(n + 1) = n + 1 \),
(3) \( \neg(\neg \varphi(0) \land \varphi(n + 1) \land \epsilon_\varphi(n) = 0) \rightarrow \epsilon_\varphi(n + 1) = \epsilon_\varphi(n) \),
(4) \( \epsilon_\varphi(n) = m + 1 \rightarrow \epsilon_\varphi(m + 1) = m + 1 \land \epsilon_\varphi(m) = 0 \).

The function \( \epsilon_\varphi \) can be defined by

\[
\epsilon_\varphi(n) = \min_{(\varphi \land k \leq n) \land \forall k = 0} (n).
\]

Using our minimum axiom, it is not difficult to prove (1)–(4). It is well known (see, for instance, [3]) that (1)–(4) imply open induction. Thus, our minimum axiom also implies it. The axiom of open induction, which in our case is a theorem, may be stated as follows:

Open Internal Induction. Let \( \varphi \) be an internal open formula, where neither \( N \) nor \( \min \) occur. Then

\[(\varphi(0) \land \forall x ((N(x) \land \varphi(x)) \rightarrow \varphi(x + 1))) \rightarrow \forall x (N(x) \rightarrow \varphi(x)).\]

We need a sort of Archimedean axiom. We introduce an operation symbol \( li \), which is internal. The expression \( li(x) \) is interpreted as "a natural number \( \geq x \)".

Axiom 3. \( N(li(x)) \land li(x) \geq x \).

We also need a maximum operator, introduced by recursion:

Axiom 4. Let \( \tau \) be a term where \( \min \) does not occur and \( x_1, \ldots, x_m \) are its distinct free variables, except for the first one. We introduce an \((n + 1)\)-ary function symbol \( \max_\tau(n, x_1, \ldots, x_m) \) with the axioms
(1) \( \max_r(1, x_1, \ldots, x_m) = 1 \),
(2) \( \mathcal{N}(n) \land \tau(n + 1) \leq \tau(\max_r(n, x_1, \ldots, x_m)) \rightarrow \max_r(n + 1, x_1, \ldots, x_m) = \max_r(n, x_1, \ldots, x_m) \),
(3) \( \mathcal{N}(n) \land \tau(n + 1) > \tau(\max_r(n, x_1, \ldots, x_m)) \rightarrow \max_r(n + 1, x_1, \ldots, x_m) = n + 1 \).
(4) \( \mathcal{N}(n) \rightarrow \mathcal{N}(\max \tau(n, x_1, \ldots, x_m)) \).

We also need the recursive definition of \( \sum \): 

**Axiom 5.** Let \( \tau(k) \) be a term where \( \min \) does not occur and \( x_1, \ldots, x_n \) are its distinct free variables, except for the first one. We introduce an \((n + 1)\)-ary function symbol \( \sum_{k=1}^{n} \tau(k) \) with the axioms:

(1) \( \sum_{k=1}^{n} \tau(k) = \tau(1) \),
(2) \( \mathcal{N}(n) \rightarrow \sum_{k=1}^{n+1} \tau(k) = \sum_{k=1}^{n} \tau(k) + \tau(n + 1) \).

With the \( \delta \) function, we can define sums restricted to certain properties. We define \( \delta_1(x) = (\delta(x) + 1)/2 \). Then

\[
\delta_1(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \tau \) and \( \sigma \) be terms. Then

\[
\sum_{i=0}^{n} \tau(i) \delta_1(\sigma(i) - x) = \sum_{\sigma(i) \geq x}^{n} \tau_i.
\]

In these last two axioms, which are axiom schemas, the new variable is \( n \). We prove the usual properties of sums and maxima by open induction, in particular, the identity axioms for max and \( \sum \). In order to be able to use open internal induction, we require that in these two axioms \( \tau(k) \) not contain \( \min \). On the other hand, since \( n \) is a variable, we may replace it by \( \min \)-terms.

For instance, we need to prove by induction a lemma about the maximum operator, which shows that it has the right meaning. From now on, the variables \( n, m, p, q, i, j, k, v, \mu, \eta \) are restricted to natural numbers, i.e., elements of \( \mathcal{N} \).

**Lemma 2.1.** Let \( \tau \) be a term where \( \min \) does not occur. Then

\[
\mathcal{N} \left( \max_{\tau}(v) \right) \land \max_{\tau}(v) \leq v \land \forall k \left( k \leq v \rightarrow \tau \left( \max_{\tau}(v) \right) \geq \tau(k) \right).
\]

The proof is by open internal induction, which, as we know, is a consequence of Axiom 2 (see [12] for a proof).

In order to develop Taylor series approximations, which we shall do in \( \S 9 \), we need to define by recursion natural number powers and factorials:

**Axiom 6.** (1) \( x^1 = x \),
(2) \( \mathcal{N}(n) \rightarrow x^{n+1} = x^n \cdot x \).

**Axiom 7.** (1) \( 1! = 1 \).
(2) \((n + 1)! = n!(n + 1)\).

While the max and \(\sum\)-axioms schemas are each an infinite collection of axioms, one for each term \(\tau\), the power and factorial axioms are particular formulas, where \(x\) is a variable, so that \(x^n\) has two variables, \(x\) and \(n\), and \(n!\) one variable. Thus, we could have terms with \(\min\) substituted for \(x\) or \(n\). We would also add other terms defined by open internal induction.

The axioms for infinitesimals are the following.

**Axiom 8.** \(\text{Inf}(x) \land \text{Inf}(y) \rightarrow \text{Inf}(x + y)\).

**Axiom 9.** \(\text{Inf}(x) \land \neg \text{Inf}(1/y) \rightarrow \text{Inf}(xy)\).

**Axiom 10.** \(x \neq 0 \land \text{Inf}(x) \rightarrow \neg \text{Inf}(1/x)\).

**Axiom 11.** \(\text{Inf}(x) \land |y| \leq |x| \rightarrow \text{Inf}(y)\).

**Axiom 12.** \(\text{Inf}(1/x) \land \neg \text{Inf}(1/y) \rightarrow \text{Inf}(1/(x + y))\).

**Axiom 13.** \(\neg \text{Inf}(1/x) \land \neg \text{Inf}(1/y) \rightarrow \neg \text{Inf}(1/(x + y))\).

**Axiom 14.** \(\neg \text{Inf}(y) \land \text{Inf}(x) \rightarrow |x| \leq |y|\).

**Axiom 15.** \(N'(v_0) \land \text{Inf}(1/v_0)\).

The constant \(e_0\) is an infinitesimal introduced by \(e_0 = 1/v_0\). It is clear that we have \(\text{Inf}(e_0)\) and \(e_0 > 0\). We introduce \(\approx\), which means "approximately equal", as a defined notion, i.e., \(x \approx y\) if and only if \(\text{Inf}(|x - y|)\). The symbol \(\approx\) is not internal. We also introduce the following expressions as definitions: \(x \approx \infty\) if and only if \(\text{Inf}(1/x) \land x \geq 0\). The expression \(x \approx \infty\), which is not internal, can be read "\(x\) is positive infinite". We write \(x \ll \infty\) for \(\neg \text{Inf}(1/x) \lor x \leq 0\). The expression \(x \ll \infty\), which is not internal, can be read "\(x\) is not positive infinite", that is, "\(x\) is nonnegative finite or negative". Similarly, we write \(x \gg y\) for \(\neg \text{Inf}(x - y) \land x \geq y\).

**Axiom 16.** Let \(\varphi\) be an open formula (not necessarily internal), where neither \(N\) nor \(\min\) occur and \(x_1, \ldots, x_n\) are the distinct free variables in \(\varphi\), except for the first variable. We introduce an \(n\)-ary function symbol \(\min_{\varphi}\), with the following axiom:

\[
\mathcal{N}(x) \land \varphi[x] \land |x| \ll \infty \rightarrow \mathcal{N} \left( \min_{\varphi}(x_1, \ldots, x_n) \right) \land \min_{\varphi}(x_1, \ldots, x_n) \leq x
\]

\[
\land \varphi \left( \min_{\varphi}(x_1, \ldots, x_n) \right).
\]

Just as for internal open induction, this axiom implies a form of Herbrand's minimum axiom and, also, a form of open external induction. Open external induction, which in our case is a theorem, may be stated as follows:

**Open External Induction.** Let \(\varphi\) be an open formula, where neither \(N\) nor \(\min\) occur. Then

\[
(\varphi(0) \land \forall x ((\mathcal{N}(x) \land |x| \ll \infty \land \varphi(x)) \rightarrow \varphi(x + 1)))
\]

\[
\rightarrow \forall x (\mathcal{N}(x) \land |x| \ll \infty \rightarrow \varphi(x)).
\]
We need to introduce derivatives and integrals at least for all elementary functions. One problem is that we cannot prove that the functions defined as inverses of other functions (such as the exponential) are defined on all numbers. The most we can prove is that for any number there is an approximately equal number where the function is defined. We must, then, complicate the definitions of derivatives and integrals to allow for this possibility. In order to have the transcendental functions defined on the right domains, we use Taylor series in §9.

The domain of inverse functions is the range of a function (for instance, in the case of the exponential, its domain is the range of the logarithm). Terms, however, are defined everywhere. So, a function in our system is determined by two terms, say $\tau$ and $\sigma$, where min does not occur, and an open formula, say $\varphi$, not necessarily internal: the argument is a value $\sigma(u)$, for a certain $u$ that satisfies $\varphi(u)$, and the value is $\tau(\sigma(u))$. If a function $f$ is represented by a pair of terms $\tau$, $\sigma$, and a formula $\varphi$, in this fashion, and $x$ is the variable for the argument of the function, we sometimes write $f(x)$ instead of $\tau(x)$ and $x \in \text{dom } f$, for $\exists u(\varphi(u) \land x = \sigma(u))$. It is clear that, in case $\sigma(u)$ is $u$ and $\varphi(u)$ is $u = u$, then the domain contains all real numbers. We allow $\tau$ and $\sigma$ to have other variables besides $x$ and $u$, but $u$ may not occur in $\tau$ and $x$ may not occur in $\sigma$. The derivative and integral of $f$ will then be associated with the pair of terms and the formula which constitute $f$. If $f$ corresponds to $\tau \sigma \varphi$, then the term $\tau(x)$ will be denoted just by $f(x)$, $\sigma(y)$ by $f_{\text{dom}}(y)$, and $\varphi(y)$ by $\varphi_f(y)$.

The expression $x \in I$, where $I$ is an interval, may be used as an abbreviation for the appropriate inequalities. For instance, if $I = [a, b]$ then $x \in I$ means $a \leq x \leq b$, and if $I = (a, b)$ then $x \in I$ stands for $a < x < b$. We shall always assume that the endpoints of $I$ are $a$ and $b$. We also use informally the subset, intersection, and union notation. For instance, $[a, b] \subset \varphi_f$ if for every $x \in [a, b]$ we have $\varphi_f(x)$. We also introduce the image of a set by a function, $x \in f(A)$ if there is a $y \in A$ such that $x = f(y)$.

We say the a term $\sigma$ is monotone on the interval $I$ if, for every $x, y \in I$, $x < y$ implies $\sigma(x) < \sigma(y)$ or, for every $x, y \in I$, $x < y$ implies $\sigma(x) > \sigma(y)$. We say that a term $\sigma$ is Lipschitz on the interval $I$ if there is a finite $M$ such that for every $x, y \in I$, $x \approx y$, we have that $|\sigma(x) - c(y)| \leq M|x - y|$.

We shall use the following abbreviation. We say that $f$ is a function on the interval $I$ if the following conditions are satisfied:

1. There are finite $a_1, b_1$ such that $[a_1, b_1] \subset \varphi_f$ and $I \subseteq J$, where $J$ is the interval with endpoints $f_{\text{dom}}(a_1)$ and $f_{\text{dom}}(b_1)$.
2. The term $f_{\text{dom}}$ is monotone and Lipschitz on $[a_1, b_1]$.
3. For all $x, y \in [a_1, b_1]$, if $x \neq y$ then $f_{\text{dom}}(x) \neq f_{\text{dom}}(y)$.

We shall also use the letters $g$ and $h$ for functions in the sense introduced above.

Let $c \leq d$, $c, d$ finite, $v$ an infinite natural number, $du = (d - c)/v$, and $u_i = c + i du$, $[c, d] \supseteq [a, b]$. So the $u_i$, for $0 \leq i \leq v - 1$, form a partition of the interval $[c, d]$, what we call a geometric subdivision of $[a, b]$ of order $v$. We always assume that $c$ and $d$ are finite and $v$ is infinite. Notice that a geometric subdivision is determined by three numbers: the endpoints of the interval, $c, d$, and the infinite natural number $v$. So when we informally quantify over geometric subdivisions, we are formally quantifying over three numbers.
Let $f$ be a function on $[a, b]$ and $u$ the geometric subdivision of $[c, d] \supseteq [a, b]$ of order $v \approx \infty$. We shall call $v$ a selector for $f$ and $u$ on $[a, b]$ if $v$ is a geometric subdivision of $[a_1, b_1]$ such that for every $i$ such that $a < u_i < u_{i+1} < b$, there are $j$'s with $a_i \leq v_j \leq b_1$ and $f_{\text{dom}}(v_j) \in [u_i, u_{i+1}]$.

Recall that the min terms can be avoided in this situation. We have the following proposition, which is one of the reasons for introducing functions this way:

**Proposition 2.2.** Let $f$ be a function on $[a, b]$. Let $u$ be a geometric subdivision of $[a, b]$. Then there is a selector for $f$ and $u$ on $[a, b]$.

**Proof.** Let $M$ be the finite number in the definition of Lipschitz for the interval $[a, b]$. Let $\varepsilon < du, b - u_{\text{min}_{uk} \geq b - 1}, u_{\text{min}_{uk} \geq a} - a$. Assume that $f_{\text{dom}}$ is increasing. Notice that

$$u_{\text{min}_{uk} \geq b} = \sum_{i=1}^{n} \delta_1(u_i - b)\delta_1(-\delta(u_i - 1 - b))u_j,$$

and similarly for $u_{\text{min}_{uk} \geq a}$, so that we can define $\varepsilon$ without min. Let $v_j = a_1 + j\varepsilon/M, v_1 = (M(b_1 - a_1)/\varepsilon, and d = a_1 + v_1\varepsilon/M$. We prove that for every $i$ such that $a \leq u_i < b$, there is a $j$ such that $f_{\text{dom}}(v_j) \in [u_i, u_{i+1}]$. In fact, we take $j = \min_{f_{\text{dom}}(v_k) \geq u_i}$. This $j$ works, because, since $f_{\text{dom}}$ is increasing $f_{\text{dom}}(v_{j-1}) < u_i$ and

$$|f_{\text{dom}}(v_j) - f_{\text{dom}}(v_{j-1})| \leq M|v_j - v_{j-1}| \leq M\frac{\varepsilon}{M} = \varepsilon \leq u_{i+1} - u_i.$$

If $f_{\text{dom}}$ is decreasing, the proof may be obtained similarly.

In order to prove the second conclusion, we take $u$ to be the geometric subdivision of $[c, d]$ of order $v_0$. Let $x \in [a, b]$ and $n = \min_{uk \geq x}$. We have that $d = u_n \gg x$. Hence, by Axiom 2, $u_n \geq x$. If $n = 0$, then $u_n = c \leq x$, and, hence $u_n = x$. If $n > 0$, then, again by Axiom 2, $x > u_{n-1}$. But $u_n \approx u_{n-1}$, and so $u_n \approx x$. Therefore, in any case, $u_n \approx x$. By the previous part of the proposition, there is an $f_{\text{dom}}(y)$ in $[u_n, u_{n+1}]$. But $f_{\text{dom}}(y) \approx u_n \approx x$.

Notice that, given $M$, we can construct that $j$ such that $f_{\text{dom}}(v_j) \in [u_i, u_{i+1}]$.

By the differential of $f$, $df(x, y)$, we mean the difference $f(x + y) - f(x)$. We now introduce the derivative: For each pair of terms, $\tau(x, x_1, \ldots, x_n)$ and $\varphi(y, y_1, \ldots, y_m)$, each open formula $\varphi(y, z_1, \ldots, z_p)$, where the variables are as displayed, and each variable $x$, we introduce the $(n + m + p + 1)$-ary operation $(\tau \varphi)_x$. If $\tau$ is written as the function $f(x)$, we write $f'(x)$ for the term $(\tau \varphi)_x(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_p)$.

**Axiom 17.** Let $f$ be a function on $I$, where $|a|, |b| \ll \infty$. Then, if for every $x, y$ such that $\inf(y), y \neq 0, x + y \in I \cap \text{dom } f$, we have that $df(x, y)/y \approx g(x)$,

then, for every $x \in I \cap \text{dom } f$, $f'(x) \approx g(x)$.

For the integral, we assume that $f$ is the function on $I$ determined by $\tau, \varphi$ as above. Then, we introduce the $(n + m + p + 2)$-ary operation symbol

$$\int_I^\psi (\tau \varphi) dx(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_p),$$
which is called the integral of $f$ with respect to $x$. We also write $\int_t^u (\sigma \delta) \, dx$ as $\int_t^u f$.

Before we state the corresponding axiom, we will introduce a few abbreviations and explanations. Let $f$ be as above. We shall approximate definite integrals on the interval $[a, b]$ by sums, that is, by the nonstandard counterpart of Riemann sums. Since we may have the function $f$ undefined on part of the interval $[a, b]$, but only defined on the range of another function represented by the term $f_{\text{dom}}$, we need to be able to choose points in each interval $[u_i, u_{i+1}]$ where $f$ is defined. We have seen in Proposition 2.2 that this is always possible when $f$ is a function on $[a, b]$. But this was done in the proof with min-terms, and we cannot use min-terms inside sums. Fortunately, since $f_{\text{dom}}$ is monotone, we can choose this number without min-terms:

Let $v$ be a selector for $f$ and $u$ on $[a, b]$, which is a geometric subdivision of the interval $[a_1, b_1]$ of order $v_1$. Then, for every $i < \nu$ there is a $v_j$ such that $f_{\text{dom}}(v_j)$ is in the interval $[u_i, u_{i+1}]$. Let the functions $\delta$ and $\delta_1$ be as introduced before. If $f_{\text{dom}}$ is strictly increasing, then the term

$$
\sum_{j=0}^{v_1-1} \delta_1(f_{\text{dom}}(v_j) - u_i) \delta(u_{i+1} - f_{\text{dom}}(v_j)) \times \delta_1(-\delta(f_{\text{dom}}(v_{j-1}) - u_i)) f(f_{\text{dom}}(v_j)) = f(f_{\text{dom}}(v_k)),
$$

where $k = \min f_{\text{dom}}(v_k) \in [u_i, u_{i+1}]$. This can be easily proved using Axiom 2. For strictly decreasing terms, we have to change $v_{j-1}$ to $v_{j+1}$ and then get $k$ as the last $j < \mu$ such that $f_{\text{dom}}(v_j) \in [u_i, u_{i+1}]$, i.e., $k = \min f(v_k) \in [u_{i+1}, u_{i+2}]$. This also is proved using Axiom 2. We denote $f_{\text{dom}}(v_k) = t_i$. We also need to find a value $t_a = f_{\text{dom}}(v_j)$ in $[a, u_{\min u_k > a}]$ and $t_b = f_{\text{dom}}(v_j)$ in $[u_{\min u_k > b} - 1, b]$. The formula is similar with the obvious replacements.

We now define the “Riemann sums”. We need six parameters: $c, d, c_1, d_1, v$, and $\mu$, which in fact may be summarized as two geometric subdivisions. If $u$ is the geometric subdivision of $[c, d] \supseteq [a, b]$ of order $v$ and $v$ is a geometric subdivision of $[c_1, d_1]$ of order $v_1$, which is a selector for $f$ and $u$ on $[a, b]$, we abbreviate

$$
\sum_a^b f[u, v] = \sum_{t_i}^v f(t_i) \, du.
$$

Recall that $f(t_i)$ is an abbreviation for the term

$$
\sum_{j=0}^{\mu-1} \delta_1(f_{\text{dom}}(v_j) - u_i) \delta_1(u_{i+1} - f_{\text{dom}}(v_j)) \times \delta_1(-\delta(f_{\text{dom}}(v_{j-1}) - u_i)) f(f_{\text{dom}}(v_j)),
$$
when $f_{\text{dom}}$ is strictly increasing, and for the term
\[
\sum_{j=0}^{\mu-1} \delta_1(f_{\text{dom}}(v_j) - u_j) \delta(u_{i+1} - f_{\text{dom}}(v_j)) \times \delta_1(-\delta(f_{\text{dom}}(v_{j+1}) - u_j)) f(f_{\text{dom}}(v_j)),
\]
when $f_{\text{dom}}$ is strictly decreasing. We are now ready to introduce our axiom. We use the notation introduced above.

**Axiom 18.** Let $f$ be a function on $[a, b]$. Suppose that $a \leq b < c$, $|a|, |c| \ll \infty$. Under these hypotheses we have that the condition:

If for all geometric subdivisions $u$, $u'$ of $[a, c]$ such that $\frac{du}{c-b} \approx 0$, $\frac{du'}{c-b} \approx 0$, and all selectors $v$ for $f$ and $u$ on $[a, c]$ and $v'$ for $f$ and $u'$ on $[a, c]$ we have
\[
(1) \sum_{c-b}^b f(u, v) + \sum_{c-b}^c f(u, v) \approx \sum_{c-b}^c f(u, v),
\]
and
\[
(2) \sum_{c-b}^c f(u, v) \approx \sum_{c-b}^d f(u', v'),
\]
implies that, for every geometric subdivision $u$ of $[a, c]$, such that $\frac{du}{c-b} \approx 0$, and every selector $v$ for $f$ and $u$ on $[a, b]$, we have
\[
(a) \frac{f_a}{c-b} f + \frac{f_b}{c-b} f \approx \frac{f_c}{c-b} f,
\]
and
\[
(b) \frac{f_c}{c-b} f \approx \frac{f_d}{c-b} f.
\]

§3. Consistency. We can use familiar results in nonstandard analysis (see, for instance, [5], [1, Chapter 1], or [2, Appendix A]) to establish the validity of our axioms in a system of nonstandard analysis and thus show the relative consistency with classical analysis (and ZFC) of our system of axioms when the function symbols have the usual interpretation, members of $^*\mathbb{R}$ are assigned to numerical variables, Inf$(x)$ is interpreted as $x$ is an infinitesimal, and $N'(x)$, as $x \in ^*\mathbb{N}$. The proof consists of easy verifications. Notice that we have neither the notion of standard functions nor the standard part function, so that we cannot characterize real numbers and all our functions are internal (but cannot be proved standard), in the usual nonstandard sense.

In the rest of this section, we give a finitary proof of consistency, using Herbrand's theorem. We first show that the theory with the axioms for derivatives (Axiom 17) and for integrals (Axiom 18) is a conservative extension of the theory without them. Suppose we have a proof of a statement $\varphi$ where $'$ does not occur in which there is a line $\psi_i$ with an instance of Axiom 17. In the conclusion we have that $x \in \text{dom} f$ and $x \in I$, i.e., $x = f_{\text{dom}}(y)$ for a certain $y \in [a_1, b_1]$. Let $z = f_{\text{dom}}(y + \varepsilon_0) - f_{\text{dom}}(y)$. Then $z \approx 0$, but $z \neq 0$. Replace $f'(x)$ by $df(x, z)/z$ in $\psi_i$. Then the new statement is derivable from the infinitesimal axioms. It is easy to see that the minimum and maximum schemas are transformed by this substitution into schemas of the same kind.

We may notice that we could have introduced, instead of Axiom 17, a conditional definition of $f'$ by putting $f'(x) = df(x, z)/z$ instead of $f'(x) \approx g(x)$ in the conclusion of the axiom. We did not do this because, although it would have
simplified the system, the standard derivative of a standard function $f$ is not, in
general, $df(x, z)/z$. Since we do not use standard functions, this would not have
been a problem in our system.

The argument for Axiom 18 is similar, but somewhat more complicated. Let
$u$ be the geometric subdivision of $[e_1, e_2] \supseteq [a, c]$ of order
$$\lim_{v_0} \left( \frac{e_2 - e_1}{c - b} \right).$$

Then
$$\frac{du}{c - b} = \frac{e_2 - e_1}{v(c - b)} \leq \frac{1}{v_0} \approx 0.$$  

Find $v_1$, $a_1$, and $d_1$ as in the proof of Proposition 2.2, determining a selector $v$
for $f$ and $u$ on $[a, c]$. In the conclusion of the axiom replace $\int_a^b f$ by $\sum_a^b f[u, v]$, $\int_a^c f$ by $\sum_a^c f[u, v]$, and $\int_b^c f$ by $\sum_b^c f[u, v]$. The new line is a consequence of
the infinitesimal axioms.

Thus, by eliminating the axioms for derivatives and integrals, we reduce the
theory to the theory of a field, which may be taken to consist of algebraic numbers,
plus infinitesimals.

**Construction of the finite model.** We know from Herbrand's theorem that the
theory is inconsistent if and only if there is a conjunction of closed substitution
instances of the axioms which is inconsistent. We also have to include instances of
the equality axioms for the field operations. The corresponding equality theorems
for the operations defined by recursion are proved by induction, so that we do not
need to include their instances. We consider a particular conjunction of substitu­
tion instances, say $\psi$, of the axioms with constant terms. We use the expression
the terms in the instance for terms occurring in the substitution instance $\psi$.

We shall construct a model, for every instance $\psi$, whose universe is a finite set
of rational numbers, and for which the field operations and relations have their
natural meanings. Also, natural numbers will be represented by natural numbers
in the model. We thus show the consistency of $\psi$. As we shall see, the instances
of the equality axioms for the field operations will be automatically satisfied, so
we do not need to worry about them. The models for the instances $\psi$ are sort
of finite approximations or fragments of the nonstandard fields constructed by A.
Robinson in [9].

The model is constructed in two steps. We first associate to each of the constant
terms in the instance, say $\tau$, an expression $\tau^*$ of the form

$$a_0 v_0^{r_0} + a_1 v_0^{r_1} + \cdots + a_n v_0^{r_n}$$

$$b_0 v_0^{s_0} + b_1 v_0^{s_1} + \cdots + b_m v_0^{s_m},$$

where $a_0, \ldots, a_n, b_0, \ldots, b_m$ are positive rational powers of rational numbers and
$r_0, \ldots, r_n, s_0, \ldots, s_m$ are specific nonnegative rational numbers. We assume that
$r_0 < r_1 < \cdots < r_n$ and $s_0 < s_1 < \cdots < s_m$. The numbers $n$ and $m$ are specific
natural numbers, not variables. This association will have the properties that the
* operation commutes with the field operations and preserves the equalities and
inequalities. For brevity, we shall call functions of the form $(\star)$ algebraic functions
of \( v_0 \). In the second step, we associate a rational number with each term. The universe of the model is a finite set of rational numbers.

The association will have the property that \( \tau = \sigma \) can be proved with the field axioms if and only if \( \tau^* = \sigma^* \), so that \( \tau^* \) is a sort of normal form of \( \tau \). Thus, the equality axioms for the field operations are satisfied.

The construction of the model is based on the well-known fact that if \( r_n > s_m \) then an algebraic function of \( v_0 \) tends to \( \pm \infty \), if \( s_m > r_n \) it tends to \( 0^+ \) or \( 0^- \), and if \( r_n = s_m \) it tends to \( a_n/b_m \), when \( v_0 \) tends to \( +\infty \). These limits can be constructively approximated.

It is to be noted that algebraic numbers that are not rational, and rational powers that are not integer powers appear only to eliminate the minimum operator. Without this operator, we only get rational functions of \( v_0 \).

In the first place, given the field axioms and taking \( \varepsilon_0 = 1/v_0 \), all terms are equal to rational functions of \( v_0 \). That is, a term is equal to

\[
\frac{a_0 + a_1 v_0 + \cdots + a_n v_0^n}{b_0 + b_1 v_0 + \cdots + b_m v_0^m},
\]

where \( a_0, \ldots, a_n, b_0, \ldots, b_m \) are terms constructed from 0, 1, \( \delta, \) min, and max terms, \( \Sigma \)-terms and natural number powers. Notice that the terms inside the \( \delta \) function, \( \min, \max, \sum, \) and powers can be set in the same form. Thus, we have to eliminate the terms with \( \delta, \min, \max, \sum, \) and powers.

The \( \Sigma \)-terms. We shall associate effectively to terms of the form

\[
\sum_{i=1}^{g(v_0)} \tau(i)
\]

an algebraic function of \( v_0 \), where \( g(v_0) \) is an algebraic function of \( v_0 \) and the \( \tau(i) \) are algebraic functions of \( v_0 \). The symbols \( \delta, \max, \sum, \) or powers may occur in \( \tau(i) \) and \( g(v_0) \), and \( \min \) may occur in \( g(v_0) \), so that the transformation to algebraic functions of \( v_0 \) must be done recursively.

If a \( \Sigma \)-term does not occur in an instance of Axiom 5, the sum axiom, associate an arbitrary number. In case an instance of Axiom 5(1) occurs

\[
\sum_{i=1}^{1} \tau(i) = \tau(1),
\]

we set

\[
\left( \sum_{i=1}^{1} \tau(i) \right)^* = \tau(1)^*.
\]

On the other hand, let \( m, m+1, \ldots, m+q \) be a maximal chain of terms in normal form from the instance \( \psi \) of the form

\[
\sum_{i=1}^{m+j+1} \tau(i) = \sum_{i=1}^{m+j} \tau(i) + \tau(m+j+1).
\]
Then we set
\[
\left( \sum_{i=1}^{m} \tau(i) \right)^* = \tau(m)^*,
\]
and we let
\[
\left( \sum_{i=1}^{m+j+1} \tau(i) \right)^* = \left( \left( \sum_{i=1}^{m+j} \tau(i) \right)^* + \tau(m+j+1) \right)^*.
\]

The max-terms. We proceed similarly as for the \( \sum \)-terms. The term \((\max_r(1))^* = 1 \). On the other hand, let \( m, m+1, \ldots, m+q \) be a maximal chain of terms in normal form from the instance \( \psi \) occurring in formulas of \( \psi \) of the form
\[
\mathcal{N}(m+j) \land \tau(m+j+1) \leq \tau \left( \frac{\max(m+j)}{\tau} \right) \to \max(m+j+1) = \max(m+j),
\]
or
\[
\mathcal{N}(m+j) \land \tau(m+j+1) > \tau \left( \frac{\max(m+j)}{\tau} \right) \to \max(m+j+1) = m+j+1.
\]
We then set \((\max_r(m))^* = m \). Suppose that \((\max_r(m+j))^* \) has been set. Take \( \mu \) large enough so that for all \( \nu_0 \geq \mu \),
\[
\tau(m+j+1) \leq \tau \left( \frac{\max(m+j)}{\tau} \right)^*
\]
or, for all \( \nu_0 \geq \mu \),
\[
\tau(m+j+1) > \tau \left( \frac{\max(m+j)}{\tau} \right)^*.
\]
In the first case, take \((\max_r(m+j+1))^* \) to be \((\max_r(m+j))^* \), and, in the second, \( m+j+1 \).

Other recursive definitions. The procedure is similar as for the \( \sum \) and max-terms. Take, for instance, the natural powers axiom, Axiom 6. We assume that \( m, m+1, \ldots, m+q \) is a maximal chain of terms in normal form from the instance \( \psi \) occurring in formulas of \( \psi \) of the form \( \tau^{n+1} = \tau^n \tau \), for a certain constant term \( \tau \), already in normal form. Then set \((\tau^n)^* = \tau \) and \((\tau^{n+1})^* = ((\tau^n)^* \tau)^* \).

The \( \delta \)-terms. We shall find, in this case, a natural number \( \mu \) so that these terms have constant values for all \( \nu_0 \geq \mu \). Suppose that \( \delta(\tau) \) occurs in the instance, where \( \tau \) is an algebraic function of \( \nu_0 \). Then there is a \( \mu \) such that \( \tau \geq 0 \) for all \( \nu_0 \geq \mu \), or \( \tau < 0 \) for all \( \nu_0 \geq \mu \). Let \( \delta(\tau)^* \) be 1 or -1, accordingly.

The min- and li-terms occur in instances of Axioms 3, 2, or 16. If a min- or li-term \( \tau \) does not occur in an instance of one of these axioms, set \( \tau^* = 1 \).

If general, if a min- or li-term \( \sigma \) occurs in one of these axioms, we shall take \( \sigma^* \) to be either a definite natural number or an approximation of the number we shall get when \( \nu_0 \) tends to infinity. More precisely, in the latter case, if \( n(\nu_0) \) is the actual minimum for \( \nu_0 \), then \( n(\nu_0)/\sigma^* \) we tend to 1, when \( \nu_0 \) tends to infinity.
The least integer terms. We now deal with Axiom 3. Suppose that the instance is
\[
\text{li}(\tau) \geq \tau \land \mathcal{M}(\text{li}(\tau)).
\]
We proceed by induction, so that we assume that \( \tau \) does not contain min- or li-terms. We also assume that \( \tau \) is in normal form, say as \( \tau^* \). If \( \tau^* \) tends to \(-\infty\) when \( \nu_0 \to \infty \), then set \( \text{li}(\tau)^* = 1 \). If \( \tau^* \) tends to a number \( r \), set \( \text{li}(\tau)^* \) equal to the least natural number \( \geq r \). If it tends to \(+\infty\), then set \( \text{li}(\tau)^* = \tau^* \). This last setting is a good approximation, because if \( n \) is the least natural number \( \geq \tau^* \), then \( n \geq \tau^* \geq n - 1 \), and thus, \( n/\tau^* \) tends to 1 as \( \nu_0 \) tends to \( \infty \).

The min-terms. We now proceed to the min-terms. We assume that our min-term occurs in an instance of Axiom 2 or 16. We first have to define a formula \( \varphi^* \) for each formula \( \varphi \) occurring in one of the instances of the min-axioms in \( \psi \). Let min\(\varphi \) occur in an instance of Axiom 2 or 16 in \( \psi \), where \( \varphi(k, x_1, \ldots, x_n) \) is an open formula, and assume that no \( \mathcal{M} \) or min occurs in \( \varphi \) and that its only distinct variables are \( k \) and \( x_1, \ldots, x_m \). Assume that \( \varphi(n, \sigma_1, \ldots, \sigma_m) \) occurs in the antecedent of the axiom, where \( n \) and \( \sigma_1, \ldots, \sigma_m \) are constant terms in normal form (i.e., the transform * of a term). Terms with min may occur only in \( n, \sigma_1, \ldots, \sigma_m \).

We have to determine an approximation for \( \min\varphi(\sigma_1, \ldots, \sigma_m) \), i.e., for the minimal \( k \) (or minimal finite \( k \), for the external min-axiom) that satisfies \( \varphi(k, \sigma_1, \ldots, \sigma_m) \). We write \( \varphi(k, \sigma_1, \ldots, \sigma_m) \) as \( \varphi(k, \alpha_1, \ldots, \alpha_p) \), where \( \alpha_1, \ldots, \alpha_p \) are the maximal distinct min-constant terms, and replace them by distinct new variables to obtain \( \varphi(k, y_1, \ldots, y_p) \). Notice that in \( \varphi(k, y_1, \ldots, y_p) \) there are no min-terms.

For any formula \( \theta \), denote by \( \theta^* \) the formula obtained by replacing \( \tau \) by \( \tau^* \), for any term \( \tau \) occurring in \( \theta \). We define \( (\varphi(k, y_1, \ldots, y_p))^* \) so that the formula obtained by substituting \( n^*, \alpha_1, \ldots, \alpha_p \) for \( k, y_1, \ldots, y_p \) in \( (\varphi(k, y_1, \ldots, y_p))^* \) is \( (\varphi(n, \alpha_1, \ldots, \alpha_p))^* \). For instance, if the term \( \tau(n) \) is \( \sum_{i=1}^{h(n)} \sigma(i) \), and \( h(n) \) is \( \tau_1(n) + 1 \), for a certain term \( \tau_1 \), then \( \tau(n)^* \) is

\[
\left( \sum_{i=1}^{\tau_1(n)} \sigma(i) \right)^* + \sigma(\tau_1(n) + 1)^*.
\]

We take \( \tau(k)^* \) to be

\[
\left( \sum_{i=1}^{h(k)-1} \sigma(i) \right)^* + \sigma(h(k))^*.
\]

All other replacements may be done similarly.

We then order the min-terms that occur in instances of these axioms in \( \psi \), as \( \tau_1, \tau_2, \ldots, \tau_p \) in such a way that if \( i < j \), then \( \tau_j \) does not occur in \( \tau_i \). The setting of \( \tau_i^* \) is done by recursion on \( i \). The recursion will be explained later. We first consider the cases where no min-terms occur in \( \tau_i \), i.e., the formula \( \varphi \) in the corresponding axiom can be written \( \varphi(k) \), with no \( y_1, \ldots, y_p \). In the antecedent of the axiom we have \( \varphi(n) \), and a min-term may occur in \( n \). We have to consider two cases: internal and external min-terms.
The internal min-terms. In this case, the formula \( \varphi \), as above, occurring in the instance of the axiom is internal, i.e., it does not contain \( \text{Inf} \). By the field axioms, \( \varphi(k)^* \) is equivalent to a disjunction of conjunctions of formulas of the forms \( \tau > 0 \), \( \tau < 0 \), \( \tau \geq 0 \), and \( \tau \leq 0 \), where \( \tau \) is an algebraic function of \( v_0 \) with the coefficients functions of \( k \). This equivalence can be shown by noticing that: \( \tau < 0 \land \tau \neq 0 ; \tau < 0 \to \tau \geq 0 ; \tau > 0 \to \tau \leq 0 ; \tau \leq 0 \to \tau > 0 ; \) and \( \tau \geq 0 \to \tau < 0 \). These forms may be reduced to two: \( \tau > 0 \) and \( \tau \geq 0 \), by multiplying by \(-1\), if necessary.

Let us call an algebraic polynomial an expression of the form \( a_n v_0^{t_n} + \cdots + a_0 v_0^{t_0} \), where \( t_0, \ldots, t_n \) are nonnegative rational numbers and \( a_0, \ldots, a_n \) are nonnegative rational powers of rational numbers. We define similarly algebraic polynomials in several variables. We can then simplify further the terms \( \tau \). If \( \tau = \tau'/\tau'' \), where \( \tau' \) and \( \tau'' \) are algebraic polynomials, \( \tau \geq 0 \) can be replaced by \((\tau' \geq 0 \land \tau'' > 0) \land (\tau' \leq 0 \land \tau'' < 0) \). Thus, in the end, \((\varphi(k, \alpha_1, \ldots, \alpha_p))^* \) can be taken to be a disjunction of conjunctions of formulas of the forms \( \tau \geq 0 \) and \( \tau > 0 \), where \( \tau \) is an algebraic polynomial in \( v_0 \) and \( k, \alpha_1, \ldots, \alpha_p \). In order to define the minimum for such disjunction of conjunctions formulas we first determine the minimum for each atomic formula of the forms \( \tau > 0 \) and \( \tau \geq 0 \). The procedure is the same for both types of formulas, so we consider only \( \tau(k) \geq 0 \). We prove by induction that, for large \( v_0 \), the expression \( \tau(k) \geq 0 \) is equivalent to an expression of the form

\[
P_n(k) v_0^{t_n} + P_{n-1}(k) v_0^{t_{n-1}} + \cdots + P_0(k) v_0^{t_0} \geq 0,
\]

where \( P_i(k) \) is an algebraic polynomial in \( k \). We have two main cases, and one special case. The main cases are

1. \( P_n(k) > 0 \), for certain \( k \),
2. \( P_n(k) < 0 \), for all \( k \),

and the special case is that \( P_n(k) = 0 \), for certain \( k \). The special case will be reduced to the two main cases. Notice that it can be determined effectively which case we are in: take \( k_0 \) large enough so that \( P_n(k) \) has a constant sign for all \( k \geq k_0 \). Then examine all \( k < k_0 \), which are finitely many.

Suppose (case (1)) that there is a natural number \( k \) such that \( P_n(k) > 0 \). One can find effectively a minimum such \( k \), say \( k_0 \). For this \( k_0 \), there is a \( \mu \) such that \( \tau(k_0) > 0 \) for all \( v_0 \geq \mu \), and \( \tau(m) < 0 \) for all \( m < k_0 \). Set \( (\min_{\tau \geq 0})^* = k_0 \).

Suppose (case (2)) that \( P_n(k) < 0 \) for every natural number \( k \). If, for large \( v_0 \), there is no \( k \) such that \( \tau(k) \geq 0 \), set \( (\min_{\tau \geq 0})^* = 1 \).

Assume, then, that there is a minimum natural number \( k_0 \) such that \( \tau(k_0) \geq 0 \) for \( v_0 \) sufficiently large. We claim that \( k_0 \) tends to infinity when \( v_0 \) tends to infinity. Suppose not. Then there is a maximum such \( k_0 \), say \( m \). Find \( \mu \) large enough so that \( \tau(p) < 0 \) for all \( p \leq m \) and all \( v_0 \geq \mu \). Thus, \( \tau(m) < 0 \) for \( v_0 \geq \mu \), which is a contradiction, proving the claim.

Then, we proceed as follows. Divide both sides of the inequality \( \tau(k) \geq 0 \) by \(-P_n(k)\). We get the equivalent inequality

\[
-v_0^{t_n} - \frac{P_{n-1}(k)}{P_n(k)} v_0^{t_{n-1}} - \cdots - \frac{P_0(k)}{P_n(k)} v_0^{t_0} \geq 0.
\]
We have three cases for the algebraic functions \(-P_i(k)/P_n(k)\) of \(k\), for \(i = 0, \ldots, n-1\), as \(k\) tends to \(\infty\). The algebraic function may tend to \(\infty\), to a finite nonnegative number, or to a negative number or \(-\infty\). We need to compensate those terms which become negative for large \(k\) with the terms which become positive. However, the indices \(i\) in the second case may be disregarded, because, if \(-P_i(k)/P_n(k)\) tends to a finite nonnegative number, we must have \(i < n\) and, since \(t_i < t_n\),

\[
-v_0^{t_i/n} - \frac{P_i(k)}{P_n(k)} v_0^{t_i} \to -\infty,
\]
as \(v_0\) tends to \(\infty\). It is clear that, if there are no \(i\) with \(-P_i(k)/P_n(k) \to \infty\) as \(k \to \infty\), then \(\tau < 0\) for all large enough \(v_0\). Thus, we need at least one term which tends to \(\infty\). We then consider the cases where \(-P_i(k)/P_n(k)\) tends to \(\infty\) or becomes negative, and we assume that it tends to \(\infty\), for at least one \(i\), \(0 \leq i < n\).

Recall that one of the terms that tends to \(\infty\) must be made larger than or equal to all terms becoming negative.

Let \(m_1, m_2, \ldots, m_j\) be such that

\[
-\frac{P_{m_i}(k)}{P_n(k)} \to \infty \text{ of the same order as } \frac{a_i}{b_i} k^{p_i},
\]
for \(i = 1, \ldots, j\), where \(k\) tends to \(\infty\), and let \(q_1, q_2, \ldots, q_l\) be such that

\[
-\frac{P_{q_s}(k)}{P_n(k)} \text{ becomes negative of the same order as } \frac{c_s}{d_s} k^{h_s},
\]
for large \(k\) and \(s = 1, \ldots, l\).

In the next few statements, we write \(\simeq\), in an informal manner, for approximately equal for large \(v_0\). In order to compensate all negative terms \((s = 1, \ldots, l)\) with the \(m_i\) positive terms, we must have

\[
-\frac{P_{m_i}(k)}{P_n(k)} v_0^{t_{m_i}} \simeq -\frac{P_{q_s}(k)}{P_n(k)} v_0^{t_{q_s}},
\]
for large \(v_0\). That is,

\[
\frac{a_i}{b_i} k^{p_i} v_0^{t_{m_i}} \simeq \frac{c_s}{d_s} k^{h_s} v_0^{t_{q_s}}.
\]

Hence, if \(p_i > h_s\),

\[
\frac{a_i}{b_i} d_s \frac{1}{c_s} k^{p_i - h_s} \simeq v_0^{t_{q_s} - t_{m_i}},
\]
and, if \(p_i < h_s\),

\[
\frac{b_i}{a_i} c_s \frac{1}{d_s} k^{h_s - p_i} \simeq v_0^{t_{m_i} - t_{q_s}}.
\]

Let

\[
A_{js} = \left(\frac{b_i c_s}{a_i d_s}\right)^{(t_{m_i} - t_{q_s})/(h_s - p_i)}
\]
in the first case, and
\[ A_{ls} = \left( \frac{a_i d_s}{b_i c_s} \right)^{(t_{gs} - t_{m_i})/(p_i - h_s)} \]
in the second case. So, for an \( r \) such that \( k \simeq v_0^\prime \) (maybe except for a coefficient) will compensate all \( q_s \) against a certain \( m_i \), We need
\[ r p_i + t_{m_i} \geq r h_s + t_{qs}, \]
for \( s = 1, \ldots, l \). That is
\[
\begin{cases}
    r \geq \frac{t_{qs} - t_{m_i}}{p_i - h_s} & \text{if } p_i > h_s, \\
    r \leq \frac{t_{m_i} - t_{qs}}{h_s - p_i} & \text{if } p_i < h_s,
\end{cases}
\]
for \( s = i, \ldots, l \). If there is no \( r \) that satisfies all these inequalities, then the term \( m_i \) cannot compensate all \( q_s \). If there is such an \( r \), take \( r_i \) to be the maximum of the terms \( (t_{gs} - t_{m_i})/(p_i - h_s) \) with \( p_i > h_s \). Notice that \( r_i > 0 \), because the first negative term is \( -v_0^\prime n \), that is, \( q_1 = n \); but \( t_n \) is larger than all the \( t_{m_i}, i = 1, \ldots, j \).
Also, \( h_1 = 0 \). Now, \( p_i > 0 \), since \( -P_{n_i}(k)/P_n(k) \to \infty \). Therefore
\[ \frac{t_{q_1} - t_{m_i}}{p_i - h_1} = \frac{t_{q_1} - t_{m_i}}{p_i} > 0. \]

If no \( r \) exists for \( i = 1, \ldots, j \), then \( \tau < 0 \) for all \( k \), for sufficiently large \( v_0 \). So, since we assume that \( \tau \geq 0 \) for some \( k \), we take \( r \) to be the least \( r_i \) such that \( r_i \) exists. Let \( r = r_i \), for a certain \( i = 1, \ldots, j \). Then let \( A \) be the maximum of \( A_{ls} \), for \( s = 1, \ldots, l \).

If \( k_0 \) is the minimum which satisfies \( \tau \geq 0 \), then \( Av_0^\prime < k_0 \leq Av_0^\prime + 1 \), so that \( k_0/Av_0^\prime \) tends to 1 as \( v_0 \) tends to \( \infty \). Notice that, in this case, any natural number \( p > Av_0^\prime \) satisfies \( \tau(p) \geq 0 \).

We now deal with the special case where \( P_n(k) = 0 \) for certain \( k \), say \( n_0 \). If also \( P_n(k) > 0 \) for other \( k \), let \( k_0 \) be the least such. If \( n_0 > k_0 \), then take \( k_0 \) as the minimum and proceed as in case (1). If \( n_0 < k_0 \), pick \( \mu \) large enough so that the truth of \( \tau(n_0) \geq 0 \) is fixed for all \( v_0 \geq \mu \). If \( \tau(n_0) \geq 0 \), then take \( n_0 \) as the minimum; if not, \( k_0 \) is the minimum.

Now suppose that \( P_n(k) \leq 0 \) and that \( P_n(n_1) = 0, \ldots, P_n(n_d) = 0 \). We examine \( \tau(n_i) \) for \( i = 1, \ldots, d \) and find \( \mu \) large enough so that its sign is determined for all \( v_0 \geq \mu \). If \( \tau(n_i) \geq 0 \) for a certain \( i \), then \( n_i \) is the minimum. If \( \tau(n_i) < 0 \) for all \( i \) and sufficiently large \( v_0 \), then we proceed as in case (2), and determine that the minimum is asymptotically approximated by \( Av_0^\prime \).

The case for \( \tau > 0 \) can be dealt with similarly.

We have, then, that there is a large \( \mu \) such that the same minimum or minimum approximation works for all \( v_0 \geq \mu \). If we have a conjunction of formulas of the forms \( \tau \geq 0 \) and \( \tau > 0 \), we must make an analysis of the different combinations of cases in order to determine an approximation. Case (1) has to be divided into two subcases:

1a) There are only finitely many \( k \)'s such that \( P_n(k) > 0 \).

1b) \( P_n(k) > 0 \), for all sufficiently large \( k \).
In both subcases, we can take $v_0$ large enough so that $\tau \geq 0$ only for $k$'s such that $P_n(k) > 0$.

If we have a conjunction of formulas of the form $\tau(k) \geq 0$ and $\tau(k) > 0$ in case (1), then we can determine effectively whether there is a $k$ which satisfies all the formulas, and then determine its minimum. If there are formulas in cases (1a) and (2), then we can find a $v_0$ large enough so that the minimum for the formulas in case (2) (which is of the order of $v_0^r$, for some positive rational $r$) is larger than all the $k$ which satisfies the formulas in subcase (1a), and, hence, there is no number which satisfies all the formulas in the conjunction. Finally, if all the formulas are in cases (1b) and (2) (or only in case (2)), take the largest $A$ for the largest $r$ such that $A v_0^r$ is an approximation for the minimum of the formulas in case (2), and take $v_0$ large enough so that for all $k > A v_0^r$, the formulas in case (1b) are satisfied.

If we have a disjunction of conjunctions, then take as an approximation to the minimum the least that satisfies one of the conjunctions in the disjunction.

Finally, in order to obtain the * transform, in all terms in the instance $\psi$, replace $\text{min}_\varphi$ by its approximation thus determined.

The external min-terms. As was mentioned before, the elimination of external min-terms has to be done simultaneously by recursion with the internal min-terms. We assume, then, that we have an instance of Axiom 16 with $\varphi$ as the main formula (i.e., $\text{min}_\varphi$ is the main min-term). We also assume that $\varphi$ contains no min-terms and that $\sum$, $\delta$, and max have been eliminated.

We assume that the terms have been put in normal form, which are algebraic functions of $v_0$. As we did before, we replace the terms in $\varphi$ by their normal starred forms to obtain $\varphi^*$. In order to determine the truth of $\varphi^*$, which may contain Inf, declare a term $\tau^*$ infinitesimal, i.e., say that $\text{Inf}(\tau^*)$ is true, if $\tau^*$ has $s_m > r_n$, i.e., a higher degree denominator, and no min-terms.

Let $p_0$ be a natural number such that $|a_n/b_m| \leq p_0$ for all terms $a_n, b_m$ in the instance where no min-term occurs and $s_m \geq t_n$. We take $v_0$ large enough so that all inequalities in $\varphi^*(k)$ are determined when replacing $k$ by any of the numbers $1, 2, \ldots, p_0 + 1$. Examine the formula $\varphi^*(k)$ for $k = 1, 2, \ldots, p_0 + 1$. If it is true, with the conventions adopted above, for at least one of these numbers, let $\text{min}_\varphi$ be the minimum for which it is true. If it is not, let this minimum be provisionally equal to 1. Replace, then, this minimum term wherever it occurs by the corresponding number.

The recursive procedure for elimination of min-terms. Suppose, as above, that the min-terms in instances of the min-axioms in $\psi$ are $\tau_1, \ldots, \tau_p$, ordered so that $\tau_j$ does not occur in $\tau_i$ if $i < j$. We replace $\tau_i$ by $\tau^*$ definitively or provisionally. A term $\tau_i$ is provisionally replaced if it is in an external minimum axiom provisionally replaced by 1 by the procedure given above, or if a provisionally replaced term occurs in it. A term is definitively replaced if it is not provisionally replaced.

We deal with $\tau_1, \ldots, \tau_n$ by successive changes. We first obtain $p_0$ as above, and apply the procedures for the min-terms $\tau_1, \ldots, \tau_n$ (internal or eternal) outlined above. Those that are definitively replaced are eliminated from the list to obtain $\tau_{11}, \ldots, \tau_{1m_1}$. Look again at the terms of the form $a_n/b_m$ with the degree of the
numerator less than or equal the degree of the denominator, and, as above, find a natural number \( p_1 \) such that \( |a_n/b_m| \leq p_1 \) for all these terms. Declare a term \( \sigma \) infinitesimal if it contains no min-term and has a higher degree denominator. We have \( p_1 \geq p_0 \). If \( p_1 > p_0 \), repeat the procedures for external and internal min-terms with \( \tau_{11}, \ldots, \tau_{1m} \). Those that are definitely replaced are eliminated to obtain \( \tau_{21}, \ldots, \tau_{2m} \). If \( p_1 = p_0 \), stop.

Repeat the same procedure until stopping, i.e., until \( p_q = p_{q+1} \) for a certain \( q \). The procedure must stop, since, for \( p_{j+1} \) to be larger than \( p_j \), at least one of the terms \( \tau_{j1}, \ldots, \tau_{jm} \), say \( \tau_{jk} \), has to be changed. This can happen either by

(i) \( \tau_{jk} \) being an external min-term which is assigned a number greater than 1, while it was assigned 1 before, or

(ii) by a term occurring in \( \tau_{jk} \) being changed.

Thus each term can be changed only a finite number of times before being definitively replaced. Since there are only finitely many terms, the possible changes are finite, and so there must be a \( p_q = p_{q+1} \).

Notice, also, that in the case of internal min-terms, they are replaced either by natural numbers or by expressions of the form \( A\nu_0 \). In the second case, the replacement \( A\nu_0 \) is not, in general, a natural number. Since the min-terms will have to be assigned natural numbers in the model, the assignment for the model will change. This final change, however, causes no problems, because \( A\nu_0 \) will be an infinite number larger than the \( p_j \)'s, so it will not affect the assignments of the external min-terms.

The model. We now proceed to the construction of the model. We have associated, by the procedure indicated above, to each term an algebraic function of \( \nu_0 \). Let

\[
\frac{1}{p} \leq \left| \frac{a_n}{b_m} \right| \leq p,
\]

for all \( a_n, b_m \), with \( r_n = s_m \), of the algebraic functions associated to terms in the instance. Since there are only finitely many terms, it is always possible to find such a natural number \( p \). Find \( \nu_0 \) large enough so that the following conditions are met:

(1) All replacements for \( \delta \) and max can be done within an approximation of \( 1/(2p) \).

(2) All fractions \( \frac{\text{li}(\tau^*)}{\tau^*} \), when \( \tau^* \to \infty \), are closer to 1 than \( 1/(2p) \).

(3) All fractions \( \frac{\text{min}_{\nu^*}}{A\nu_0} \), when this replacement applies (for internal min-terms), are closer to 1 than \( 1/(2p) \).

(4) All algebraic functions with \( r_n > s_m \) are larger in absolute value than \( 2p \).

(5) All algebraic functions with \( s_m > r_n \) are smaller in absolute value than \( 1/(2p) \).

(6) All algebraic functions with \( r_n = s_m \) are closer to \( a_n/b_m \) than \( 1/(2p) \).

Let \( \nu(\tau^*) \) be the numerical value of the associated algebraic function \( \tau^* \), using this \( \nu_0 \). In the model, put all the natural numbers less than or equal to \( \mu_0 \), where \( \mu_0 \) is a natural number larger than the largest \( \nu(\tau^*) \). The property \( \mathcal{N} \) is interpreted as the set of natural numbers \( \leq \mu_0 \). We now define the interpretation \( I(\tau) \) of each term \( \tau \). If \( \tau \) does not contain li or min, then \( I(\tau) = \nu(\tau^*) \). If \( \tau = \text{min}_\varphi \) or \( \tau = \text{li}(\sigma) \),
then \( I(\tau) \) is the actual natural number which is the minimum of the numbers \( \leq \mu_0 \) which satisfies \( \varphi(k) \) or which is \( \geq \sigma^* \), respectively. All terms are formed with the field operations operating over min- or li-terms and terms without min or li, so we complete the assignment by determining that 
\[
I(\tau + \sigma) = I(\tau) + I(\sigma),
I(\tau \cdot \sigma) = I(\tau) \cdot I(\sigma),
I(\tau - \sigma) = I(\tau) - I(\sigma), \text{ and } I(\tau/\sigma) = I(\tau)/I(\sigma).
\]

Declare a term \( \tau \) infinitesimal, i.e., say that \( \text{Inf}(\tau) \) is true, if the associated algebraic function \( \tau^* \) has \( s_m > r_n \), i.e., a higher degree denominator and, hence, \( |I(\tau)| < 1/2p \); declare it finite if it has \( r_n = s_m \), i.e., an equal degree numerator and denominator and, hence, \( |I(\tau)| < p + 1 \); and declare it infinite if it has \( r_n > s_m \), i.e., a higher degree numerator and, hence, \( |I(\tau)| \geq 2p \). It is easy to check that all the infinitesimal axioms are satisfied:

Axiom 8. This is satisfied because the sum of two algebraic functions with higher degree denominators also has a higher degree denominator.

Axiom 9. Suppose that the algebraic function \( x^* \) for \( x \) has a higher degree denominator and \( (1/y)^* \) has a higher or equal degree numerator. Then \( y^* \) has a higher or equal degree denominator. Thus, \( (xy)^* \) has a higher degree denominator.

Axiom 10. It is clear that if \( x^* \) has a higher degree denominator, then \( (1/x)^* \) does not.

Axiom 11. Suppose that \( x^* \) has a higher degree denominator, and that \( v(|y^*|) \leq v(|x^*|) \). Then, by the selection of \( v_0 \), \( y^* \) also has a lower degree numerator.

Axiom 12. Suppose that \( x^* \) has a higher degree numerator, and \( y^* \) has a lower or equal degree numerator. Then \( x^* + y^* = (x + y)^* \) has higher degree in the numerator.

Axiom 13. If both \( x^* \) and \( y^* \) have a lower or equal degree numerator, then so has \( x^* + y^* \).

Axiom 14. If \( y^* \) has a higher or equal degree numerator and \( x^* \) has a higher degree denominator, then, by the choice of \( v_0 \), \( v(|x^*|) \leq v(|y^*|) \).

Axiom 15. It is clear that \( 1/v_0 \) has a higher degree denominator.

This completes the finitary proof of consistency.

§4. Differentials and derivatives. We now begin the development of the calculus of one variable from the axioms introduced in §2. As is customary, we use the abbreviations \( (f + g)(x), (fg)(x), \frac{f}{g}(x) \), and \( (f \circ g)(x) \).

We have introduced the derivative \( f'(x) \) of \( f(x) \) in Axiom 17 in §2. We say that \( f \) is differentiable at \( x \in \text{dom } f \) if

\[
\forall y \left( \text{Inf}(y) \land y \neq 0 \land x + y \in \text{dom } f \right)
\rightarrow |f'(x)| \ll \infty \land \frac{df(x,y)}{y} \approx f'(x)
\]

We say that the function \( f \) on \( I \) is differentiable on the interval \( I \) if and only if for all \( x, y, \) if \( \text{Inf}(y), y \neq 0, x, x + y \in I \cap \text{dom } f, \) then \( |f'(x)| \ll \infty \) and \( df(x,y)/y \approx f'(x) \).

Similarly, the function \( f \) on \( I \) is continuous on the interval \( I \) if and only if for all \( x, y \in I \cap \text{dom } f, \) if \( x \approx y, \) then \( f(x) \approx f(y) \).
THEOREM 4.1. If \( f \) is a function on \( I \) that is differentiable there, then \( f \) is continuous on \( I \).

PROOF. Assume \( x \approx y, x, y \in \text{dom } f \cap I \). Then, \( \text{Inf}(y - x) \). Hence, since \( f'(x) \) is finite,

\[
f(y) - f(x) = df(x, y - x) \approx f'(x) \cdot (y - x) \approx 0.\]

Moreover, with a slightly stronger condition than differentiability we can prove that the function is Lipschitz continuous, i.e., there is a finite \( M \) such that for every \( x, y \in I \cap \text{dom } f \), \( x \approx y \), we have that \( |f(x) - f(y)| \leq M|x - y| \). In the usual nonstandard analysis, differentiability is enough to prove the following theorem.

THEOREM 4.2. If \( f \) is a function on the interval \( I \) that is differentiable there, and if there is a finite \( N \) such that \( |f'(x)| \leq N \) for every \( x \in I \cap \text{dom } f \), then \( f \) is Lipschitz continuous on \( I \).

PROOF. Let \( x, y \in I, x \approx y \). Then

\[
\frac{|f(x) - f(y)|}{|x - y|} \approx |f'(x)|.
\]

Thus

\[
\frac{|f(x) - f(y)|}{|x - y|} \leq |f'(x)| + 1 \leq N + 1.
\]

We let \( M = N + 1 \) and obtain \( |f(x) - f(y)| \leq M|x - y| \).

We can prove the following theorem for derivatives, using Theorem 4.1 and some easily-proved algebraic properties of differentials.

THEOREM 4.3. Let \( f \) and \( g \) be differentiable on \( I \). Then:

1. \( f + g \) is differentiable on \( I \) and, for every \( x \in I, x \in \text{dom } f \), we have \( (f + g)'(x) \approx f'(x) + g'(x) \).

2. With the additional hypothesis that \( f \) and \( g \) are finite on \( I \cap \text{dom } f \), we have that \( f \cdot g \) is differentiable on \( I \) and, for every \( x \in I, x \in \text{dom } f \),

\[
(f \cdot g)'(x) \approx f(x) \cdot g'(x) + g(x) \cdot f'(x).
\]

3. With the additional hypotheses that \( f \) and \( g \) are finite on \( I \cap \text{dom } f \) and \( |g(x)| \gg 0 \), for \( x \in I \), we have that \( f/g \) is differentiable on \( I \), and for every \( x \in I \)

\[
\left(\frac{f}{g}\right)'(x) \approx \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.
\]

We can also prove the chain rule:

THEOREM 4.4 (Chain rule). If \( f \) is differentiable at \( g(x) \) and \( g \) is differentiable at \( x \), then \( f \circ g \) is differentiable at \( x \) and \( (f \circ g)'(x) \approx f'(g(x))g'(x) \).
PROOF. Assume the hypotheses of the theorem, and let \( \inf(y) \). Then
\[
dg(x,y)/y \approx g'(x),
\]
i.e., \( dg(x,y) = g'(x) + \epsilon_1 y \), where \( \epsilon_1 \approx 0 \). Hence, since \( g'(x) \) is finite, by Theorem 4.1, \( \inf(dg(x,y)) \). Therefore, if \( dg(x,y) \neq 0 \),
\[
df(g(x),dg(x,y)) \approx f''(g(x)).
\]
Thus, in any case
\[
df(g(x),dg(x,y)) = f'(g(x))dg(x,y) + \epsilon_2 dg(x,y),
\]
where \( \epsilon_2 \approx 0 \). But, by the definition of composition,
\[
df \circ g(x,y) = f(g(x + y)) - f(g(x)).
\]
On the other hand, \( g(x + y) = g(x) + dg(x,y) \). Hence,
\[
df \circ g(x,y) = f(g(x) + dg(x,y)) - f(g(x)) = df(g(x),dg(x,y)).
\]
From these equations, we get the conclusion of the theorem. \( \square \)

§5. Theorems on continuous functions. We abbreviate \( x \leq y \lor \inf(x - y) \) by \( x \preceq y \). Similarly, \( x \geq y \lor \inf(x - y) \) is abbreviated by \( x \succeq y \). For continuous functions \( f \), we use the following abbreviations: \( f(x) \approx y \) means \( \exists z (z \approx x \land z \in \text{dom} f \land f(z) \approx y) \); \( f(x) \preceq y \) means \( \exists z (z \approx x \land z \in \text{dom} f \land f(z) \preceq y) \); and \( f(x) \succeq y \) means \( \exists z (z \approx x \land z \in \text{dom} f \land f(z) \succeq y) \). If \( f \) is continuous and we have \( z \approx z' \), with \( z, z' \in \text{dom} f \), then \( f(z) \approx f(z') \), so that we can replace \( \exists \) by \( \forall \) in these definitions.

We have the approximate intermediate value theorem:

**Theorem 5.1 (IVT).** If \( f \) is a continuous function on \([a,b]\) and \( f(a) \preceq 0 \preceq f(b) \), then there is an \( x \in [a,b] \) such that \( f(x) \approx 0 \).

**Proof.** Let \( u \) be the geometric subdivision of order \( v_0 \) of \([a,b]\) and let \( v \) be a selector for \( f \) and \( u \) on \([a,b]\). Suppose that \( f_{\text{dom}} \) is increasing. (The proof for decreasing \( f_{\text{dom}} \) is similar.) Assume that \( f_{\text{dom}}(v_p) \approx a \) and that \( f_{\text{dom}}(v_q) \approx b \). If \( f(f_{\text{dom}}(v_q)) \approx 0 \) or \( f(f_{\text{dom}}(v_p)) \approx 0 \), we are done. So we may assume that \( f(f_{\text{dom}}(v_q)) \geq 0 \) and \( f(f_{\text{dom}}(v_p)) < 0 \). Let \( n = \min f(f_{\text{dom}}(v_{n-1})) \). We have \( f(f_{\text{dom}}(v_q)) \geq 0 \) and, thus, by Axiom 2, \( f(f_{\text{dom}}(v_n)) \geq 0 \) and \( 0 \leq p < n \leq q \leq v_1 \). Then \( n - 1 \geq 0 \). Suppose that \( f(f_{\text{dom}}(v_{n-1})) \geq 0 \). Then, by Axiom 2, \( n - 1 \geq n \), which is impossible. Therefore, \( f(f_{\text{dom}}(v_{n-1})) \leq 0 \leq f(f_{\text{dom}}(v_n)) \). Since \( f \) is continuous and \( f_{\text{dom}}(v_{n-1}) \approx f_{\text{dom}}(v_n) \), it follows that \( f(f_{\text{dom}}(v_n)) \approx 0 \). \( \square \)

We say that \( f(x) \) is a near maximum (minimum) for \( f \) on \([a,b]\) if for every \( y \in [a,b] \) we have \( f(y) \preceq f(x) \) (\( f(y) \succeq f(x) \)). In order to simplify the statement of this and other theorems, we introduce the following abbreviations. From now on we assume that \( u \) is the geometric subdivision of \([a,b]\) of order \( v \). For a discussion of geometric subdivisions in our sense, see [12]. If \( v \) is a selector for \( f \) and \( u \) on \([a,b]\), we take \( v' = v_{n+\min_{x \in [a,b]} x \geq a_1} - 1 \) and \( v = \min_{x \in [a,b]} - \min_{x \in [a,b]} - 1 \). Recall that
the mins in these formulas can be defined without min-terms by using sums. We
also denote \( dv'_i = f_{\text{dom}}(v'_{i+1}) - f_{\text{dom}}(v'_i) \). We have that \( dv'_i \approx 0 \).

**THEOREM 5.2.** If \( f \) is continuous on \([a, b]\) and \( a \leq x \leq b \), \( a \) and \( b \) finite, then
there is a near maximum on \([a, b]\). In fact, if \( u \) is a geometric subdivision of order \( v \approx \infty \), \( v \) is a selector for \( f \) and \( u \) on \([a, b]\), and \( n = \max_{f(f_{\text{dom}}(v'_i))}(v_2) \), then
\( f(f_{\text{dom}}(v_n)) \) is a near maximum of \( f \) on \([a, b]\).

There is a similar theorem for minima.

**PROOF.** Let \( u \) and \( v \) be as in the second part of the theorem. Then \( f(f_{\text{dom}}(v'_n)) \)
\( \geq f(f_{\text{dom}}(v'_i)) \) for all \( 1 \leq i \leq v_2 \). For any \( x \in [a, b] \cap \text{dom } f \) we have \( x \approx f_{\text{dom}}(v'_j) \)
for some \( j, 1 \leq j \leq v_2 \).

By taking \( v = v_0 \), we prove the first part of the theorem. \( \square \)

We have the following theorem on local maxima, that is, in fact, an approximate
version of Rolle's theorem.

**THEOREM 5.3 (Rolle).** Let \( f \) be a differentiable function on \([a, b]\), \( a \leq x \leq b \),
\( a \) and \( b \) finite, and let \( f(a) \approx 0 \approx f(b) \). Then there is an \( x \in [a, b] \) such that
\( f'(x) \approx 0 \).

**PROOF.** Let \( u \) be the geometric subdivision of \([a, b]\) of order \( v_0 \), \( v \) a selector
for \( f \) and \( u \) on \([a, b]\), and \( n = \max_{f(f_{\text{dom}}(v'_k))}(v_2) \). Let \( f_{\text{dom}} \) be increasing. (The
proof for \( f_{\text{dom}} \) decreasing is similar.) Then
\[
\frac{df(f_{\text{dom}}(v'_n)), dv'_n}{dv'_n} \approx f'(f_{\text{dom}}(v'_n))
\]
and
\[
\frac{df(f_{\text{dom}}(v'_n), -dv'_{n-1})}{-dv'_{n-1}} \approx g(f_{\text{dom}}(v'_n)).
\]

We have \( dv'_n, dv'_{n-1} > 0, f_{\text{dom}}(v'_n) + dv'_n = f_{\text{dom}}(v'_{n+1}), \) and \( f_{\text{dom}}(v'_n) - dv'_{n-1} =
\]
\( f_{\text{dom}}(v'_{n-1}) \). Then
\[
\frac{f(f_{\text{dom}}(v'_n) + dv'_n) - f(f_{\text{dom}}(v'_n))}{dv'_n} \leq 0
\]
and
\[
\frac{f(f_{\text{dom}}(v'_n) - dv'_{n-1}) - f(f_{\text{dom}}(v'_n))}{-dv'_{n-1}} \geq 0.
\]

But
\[
\frac{f(f_{\text{dom}}(v'_n) + dv'_n) - f(f_{\text{dom}}(v'_n))}{dv'_n} \approx f'(f_{\text{dom}}(v'_n))
\]
\[
\approx \frac{f(f_{\text{dom}}(v'_n) - dv'_{n-1}) - f(f_{\text{dom}}(v'_n))}{-dv'_{n-1}}.
\]

Thus, \( f'(f_{\text{dom}}(v'_n)) \approx 0 \). \( \square \)

From Rolle's theorem we derive an approximate version of the mean value
theorem:
Theorem 5.4 (MVT). If $b - a \gg 0$ and $f$ is a differentiable function on $(a, b)$, then there is an $x \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} \approx f'(x).$$

Proof. Let $h(x) = (b - a)(f(x) - f(a)) - (x - a)(f(b) - f(a))$ for every $x \in [a, b] \cap \text{dom } f$, with $\text{dom } h = \text{dom } f$. We have $h(a) = h(b) = 0$ and $h'(x) \approx (b - a)f'(x) - (f(b) - f(a))$ for all $x \in (a, b) \cap \text{dom } f$. We can apply Rolle's Theorem 5.3 to $h$ and obtain an $x$ such that

$$((b - a)f'(x) - (f(b) - f(a)) \approx 0.$$

Since $b - a$ is noninfinitesimal, we obtain the result.

We say that the function $f$ is nearly increasing (decreasing) on the interval $I$ if and only if for every $x, y \in I$ with $x \preceq y$ we have $f(x) \preceq f(y)$ ($f(x) \succeq f(y)$). As corollaries of MVT, we obtain:

Corollary 5.5. If $f$ is a continuous function on $[a, b]$, differentiable on $(a, b)$, and, for every $x \in (a, b)$, $f'(x) \succeq 0$ ($f'(x) \preceq 0$), then $f$ is nearly increasing (decreasing) on $[a, b]$.

Proof. Let $x, y \in [a, b]$, $x < y$, and $y - x \gg 0$. By MVT, Theorem 5.4,

$$f(y) - f(x) \approx f'(z)(y - x) \gtrless 0,$$

for a certain $z$. Thus, we have proved the conclusion for the case $y \gg x$.

On the other hand, if $x \approx y$, since $f$ is continuous, $f(x) \approx f(y)$, and, hence, $f(x) \preceq f(y)$. Thus the corollary follows also in the case $x \approx y$.

Corollary 5.6. If $f$ is a differentiable function on the interval $I$, and $f'(y) \approx 0$ for every $y \in I \cap \text{dom } f$, then $f(x) \approx f(z)$ for every $x, z \in I \cap \text{dom } f$.

Proof. If $x \approx z$, since $f$ is continuous, the result is clear. So, let $x \not\approx z$. The constant function $C_0$ is a derivative of $f$ on $I$. By MVT, Theorem 5.4, for a certain $v$

$$f(x) - f(z) \approx C_0(v)(x - z) = 0.$$

We shall need the next theorem for defining inverse functions. We say that a function $f$ is strictly increasing (decreasing) on $[a, b]$ if, for every $x, y \in [a, b] \cap \text{dom } f$, we have

1. $x < y$ implies $f(x) < f(y)$ ($f(x) > f(y)$), and
2. $x \not\approx y$ implies $f(x) \not\approx f(y)$.

Theorem 5.7. If $f$ is a differentiable function on the finite interval $(a, b)$ (that is, finite and continuous on $[a, b]$), and $f'(x) \gg 0$ ($f'(x) \ll 0$) for all $x \in (a, b) \cap \text{dom } f$, then $f$ is strictly increasing (decreasing) on $[a, b]$. Thus, if

$$y = g(x) \iff x = f(y),$$

for all $y \in [a, b] \cap \text{dom } f$, then $x_1 \approx x_2$ and $x_1 = f(y_1)$, $x_2 = f(y_2)$, for certain $y_1, y_2 \in [a, b] \cap \text{dom } f$, implies $g(x_1) \approx g(x_2)$. 


Moreover, if \( y = f(x) \), \( x \in [a, b] \cap \text{dom } f \), we have that

\[
dg(y, z)/z \approx 1/f'(x),
\]

for every \( z \approx 0 \), \( y + z = f(u) \), for a certain \( u \in [a, b] \cap \text{dom } f \).

The last two conclusions of the theorem say that the inverse of \( f \) restricted to \([a, b]\) in its domain is continuous and differentiable. We cannot prove, however, that the domain of the inverse of a function whose domain is an interval is also an interval. By the intermediate value theorem, Theorem 5.1, we can only prove that for every \( c \) in the interval between \( f(a) \) and \( f(b) \) there is an \( x \approx c \) such that the inverse is defined at \( x \).

**Proof.** We have to consider two cases. First, let \( x, y \in [a, b] \cap \text{dom } f \) with \( x \ll y \). It is clear that \( y - x \ll \infty \). By the mean value theorem, Theorem 5.4, there is a \( z \) between \( x \) and \( y \) such that

\[
\frac{f(y) - f(x)}{y - x} \approx f'(z) \gg 0.
\]

Since \( 0 \ll y - x \ll \infty \), we obtain \( f(y) - f(x) \approx f'(z)(y - x) \gg 0 \).

Second, assume that \( x < y \) with \( x \approx y \). Then

\[
\frac{f(y) - f(x)}{y - x} \approx f'(x).
\]

If \( f(y) - f(x) \leq 0 \), then

\[
\frac{f(y) - f(x)}{y - x} \leq 0,
\]

and so \( f'(x) \lessdot 0 \), contradicting \( f'(x) \gg 0 \).

We now obtain the derivative of the inverse function \( g \). Let \( y + z = f(u) \).

Then \( g(y + z) = u = x + v \approx g(y) = x \), by the first part of the theorem. Thus, \( v \approx 0 \) and \( y + z = f(x + v) \). Therefore

\[
\frac{z}{v} = \frac{y + z - y}{v} = \frac{f(x + v) - f(x)}{v} \approx f'(x),
\]

i.e., since \( f'(x) \not\approx 0 \)

\[
v/z \approx 1/(f'(x)).
\]

This is equivalent to \( dg(y, z)/z \approx 1/f'(x) \). \( \square \)

Recall that differentiable means that \( f' \) exists and is finite. For each finite natural number \( n \), defining by external induction, we say that \( f^{(n)} \) is an \( n \)th order derivative on \( I \) if and only if one can define a sequence of functions (terms) \( f^{(1)}, \ldots, f^{(n)} \) such that each is a derivative of the preceding on \( \text{dom } f \). Then the following is a theorem.

**Theorem 5.8 (Taylor).** Let \( n \) be a finite natural number and \( f^{(k)} \) a \( k \)th derivative of \( f \), for \( k = 1, 2, \ldots, n + 1 \). Assume also that the following conditions hold:

1. \( x \in [a, b] \cap \text{dom } f \Rightarrow |f(x)| \ll \infty \).
2. \( f^{(n+1)} \) is continuous on \([a, b]\).
3. \( a \leq c \leq d \leq b \) (or \( a \leq d \leq c \leq b \)) and \( c, d \in \text{dom } f \).
(4) \( a \leq x \leq b \rightarrow p(x) = f(c) + f^{(1)}(c)(x - c) + f^{(2)}(c)(x - c)^2 + \ldots + f^{(n)}(c)(x - c)^n \).

(5) \( x \in [a, b] \cap \text{dom } f \rightarrow f(x) = p(x) + R_{n,e}(x). \)

Then
\[
|R_{n,e}(d)| \leq \frac{1}{(n + 1)!} |f^{(n+1)}(e)| |d - c|^{n+1},
\]
where \( e \) is a near maximum of \( |f^{(n+1)}| \) on \([c, d]\) (or \([d, c]\)).

The proof is similar to the one in [7]. This theorem, as all our theorems, is a theorem-schema. In the case of Taylor's theorem, it may be better called a metatheorem, rather than a theorem-schema. The metatheorem would be more explicitly worded: "if one has a definition of the \( n \)th derivative for the function (i.e., the term) \( f \), then the following theorem can be proved...". Thus, it applies to cases where one can actually define, typically by a recursive definition for finite \( n \), derivatives up to order \( n + 1 \).

**Proof.** Let \( e \) be a near maximum of \( f^{(n+1)} \) on \([c, d]\), which exists because \( f^{(n+1)} \) is a continuous function on \([c, d]\). Assume, first, that \( d \geq c \), and define
\[
g(x) = \frac{1}{(n + 1)!} f^{(n+1)}(e)(x - c)^{n+1} + p(x) - f(x)
= \frac{1}{(n + 1)!} f^{(n+1)}(e)(x - c)^{n+1} - R_{n,e}(x).
\]

We have \( 0 = g(c) \approx g'(c) \approx \ldots \approx g^{(n)}(c) \), and
\[
g^{(n+1)}(x) = f^{(n+1)}(e) - f^{(n+1)}(x) \geq 0.
\]

Then, by Corollary 5.5, \( g^{(n)} \) is nearly increasing, and, hence, \( g^{(n)}(x) \gtrsim 0 \), when \( x \geq c \). But \( g^{(n-1)}(a) \approx 0 \), and, hence, \( g^{(n-1)}(x) \gtrsim 0 \). By external induction, we prove that \( g^{(k)}(x) \gtrsim 0 \) for all \( k \leq n \). Hence, \( g(d) \gtrsim 0 \), and thus
\[
R_{n,e}(d) \gtrsim \frac{1}{(n + 1)!} f^{(n+1)}(e)(d - c)^{n+1}.
\]

If we take \( e_1 \) a near minimum of \( f^{(n+1)} \) on \([c, d]\), we can prove similarly that
\[
R_{n,e}(x) \gtrsim \frac{1}{(n + 1)!} f^{(n+1)}(e_1)(x - c)^{n+1}.
\]

From these two inequalities we obtain the result for \( c < d \). The case \( d < c \) is treated similarly.

\[\Box\]

§6. Overflow and undertow. We can prove the following principles of undertow and overflow:

**Theorem 6.1 (Undertow).** Let \( \varphi \) be an internal formula, where neither min nor \( N \) occurs. Then, if there is a \( v \approx \infty \) such that for all \( \mu \approx \infty \) with \( \mu \leq v \) we have \( \varphi(\mu) \), then \( \min_{\varphi} \ll \infty \).

**Proof.** Let \( m = \min_{\varphi} \), and assume that \( m \) is infinite. Then \( m - 1 \) is also infinite. We have \( \varphi(v) \), and thus, by Axiom 2, \( \varphi(m) \) and \( m \leq v \). Hence \( m - 1 \leq v \)
and, by the hypothesis, \( \varphi(m - 1) \). Thus, by Axiom 2, \( m \leq m - 1 \), which is a contradiction.

In order to make the following and other statements more understandable, we introduce the maximum of all \( k \leq v \) such that \( \varphi(k) \):

\[
\text{Max}_\varphi(v) = v - \min_{\varphi(v-k) \land k \leq v} (v).
\]

It is easy to show from Axiom 2, that

\[
\varphi(m) \land m \leq v \rightarrow (\varphi(\text{Max}_\varphi(v)) \land m \leq \text{Max}_\varphi(v) \leq v).
\]

**Theorem 6.2 (Overflow).** Let \( \varphi \) be an internal formula where neither \( \min \) nor \( \mathcal{N} \) occurs. Then, if there is a finite \( m \) such that for all finite \( n \) with \( m \leq n \) we have \( \varphi(n) \) and \( v \approx \infty \), then \( \text{Max}_\varphi(v) \approx \infty \).

**Proof.** Let \( \mu = \text{Max}_\varphi(v) \), and assume that \( \mu \) is finite. It is clear that \( \mu \geq m \). Then \( \mu + 1 \) is also finite, and \( \mu + 1 \geq m \). Hence \( \varphi(\mu + 1) \). By the consequence of Axiom 2 for maximum stated above the theorem, \( \mu \geq \mu + 1 \), which is a contradiction.

We prove by overflow:

**Proposition 6.3.** If \( |x| \leq 1/n \) for every \( n \ll \infty \), then \( x \approx 0 \).

**Proof.** By overflow, we have that \( \mu = \text{Max}_{1/k \geq |x|}(v_0) \) is infinite. Hence, \( |x| \leq 1/\mu \approx 0 \).

§7. **Hyperfinite sums.** We now start the study of hyperfinite sums, which were introduced in §2. The development in the present section is much influenced by [8]. In order to conform with the usual notation, \( u, v, \) etc. are terms with variables for natural numbers. Thus \( u_k \) is a term \( \tau(k) \). We write \( \sum_{i=m}^v u_i \) for \( \sum_{j=0}^{v-m} u_{j+m} \).

In order to simplify the notation, we write a sum of the form

\[
\sum_{i=1}^v u_i + v_1 + v_2 + \cdots + v_n,
\]

simply as \( \sum_{i=1}^\mu t_i \), including the finitely many terms with ordinary addition in the \( \sum \)-term. Strictly speaking, this may not be possible, since the terms \( v_j \) may contain \( \min \). We shall be careful, however, that the operator \( \min \) occurs only in finitely many terms of the sum. The theorems of this and the next sections should be understood with \( \sum \)-terms interpreted in this way.

The usual properties of the sum can be proved by internal induction. We begin with the theorems on approximately equal infinite sums. For the rest of the paper, we assume that \( u_i, v_i, t_i \) are internal terms where, as was mentioned above, \( \min \) occurs in at most finitely many of them. We need the following lemma.

**Lemma 7.1.** Suppose that for all \( i \) with \( 1 \leq i \leq v \), we have \( u_i \approx 0, t_i > 0 \) and \( v_i/t_i \approx u_i \). Then

\[
\frac{\sum_{i=1}^v v_i}{\sum_{i=1}^v t_i} \approx 0.
\]
In particular,
\[ \sum_{i=1}^{v} \frac{u_{i}t_{i}}{t_{i}} \approx 0. \]

Therefore, with the additional hypotheses \(|\sum_{i=1}^{v} t_{i}| \ll \infty\), we obtain

\[ \sum_{i=1}^{v} v_{i} \approx 0 \quad \text{and} \quad \sum_{i=1}^{v} u_{i}t_{i} \approx 0. \]

**Proof.** Let \( n \) be finite. Then, since \( v_{i}/t_{i} \approx u_{i} \approx 0 \), we have \(|v_{i}| \leq (1/n)t_{i}\) for \( i \leq v \). Thus

\[ \left| \sum_{i=1}^{v} v_{i} \right| \leq \sum_{i=1}^{v} |v_{i}| \leq \frac{1}{n} \sum_{i=1}^{v} t_{i}. \]

Since \( \sum_{i=1}^{v} t_{i} \) is positive and \( n \) is arbitrary, by Proposition 6.3

\[ \sum_{i=1}^{v} \frac{v_{i}}{t_{i}} \approx 0. \]

The theorem behind the theorems for integrals is the following. A similar theorem, but proved in normal nonstandard analysis, appears in [2, Theorem IX.1].

**Theorem 7.2.** Suppose that for all \( i \) with \( 1 \leq i \leq v \) we have \( t_{i} > 0, t_{i} \approx 0, \)
\( u_{i} \approx v_{i}/t_{i}, \) and \(|u_{i}| < M \ll \infty, \) and assume that \( \sum_{i=1}^{v} t_{i} \ll \infty. \) Then

\[ \sum_{i=1}^{v} u_{i}t_{i} \approx \sum_{i=1}^{v} v_{i}. \]

**Proof.** We have

\[ \left| \sum_{i=1}^{v} u_{i}t_{i} \right| \leq \sum_{i=1}^{v} |u_{i}|t_{i} \leq M \sum_{i=1}^{v} t_{i} \ll \infty \]

so that \( \sum_{i=1}^{v} u_{i}t_{i} \), or any partial sum is finite.

We shall prove the approximate equation

\[(*) \quad \sum_{i=1 \atop u \geq 0}^{v} v_{i} \approx \sum_{i=1 \atop u \geq 0}^{v} u_{i}t_{i}. \]

Similarly, it can be shown that

\[ \sum_{i=1 \atop u < 0}^{v} v_{i} \approx \sum_{i=1 \atop u < 0}^{v} u_{i}t_{i} \]

and, since

\[ \sum_{i=1 \atop u \geq 0}^{v} v_{i} = \sum_{i=1 \atop u \geq 0}^{v} v_{i} + \sum_{i=1 \atop u < 0}^{v} v_{i} \quad \text{and} \quad \sum_{i=1 \atop u \geq 0}^{v} u_{i}t_{i} = \sum_{i=1 \atop u \geq 0}^{v} u_{i}t_{i} + \sum_{i=1 \atop u < 0}^{v} u_{i}t_{i}, \]

we obtain the conclusion of the theorem.
We now prove approximate equation (*). For any natural number \( m \), consider
\[
\sum_{i=1}^{v} u_i.
\]

Assume that \( m \) is a finite natural number. For \( 1 \leq i \leq v \) and \( u_i \geq 1/m \), we have \( v_i/t_i \approx u_i \). Since \( u_i \geq 1/m \), \( u_i \) is not infinitesimal. Then \( v_i/u_i t_i \approx 1 \), and so, since \( u_i t_i \) is finite,
\[
u_i t_i (1 - 1/n) \leq v_i \leq u_i t_i (1 + 1/n),
\]
for every finite \( n \), \( 1 \leq i \leq v \), and \( u_i \geq 1/m \). Thus
\[
\left(1 - \frac{1}{n}\right) \sum_{i=1}^{v} u_i t_i \leq \sum_{i=1}^{v} v_i \leq \left(1 + \frac{1}{n}\right) \sum_{i=1}^{v} u_i t_i.
\]

Thus
\[
1 - \frac{1}{n} \leq \frac{\sum_{i=1}^{v} v_i}{\sum_{i=1}^{v} u_i t_i} \leq 1 + \frac{1}{n},
\]
and, hence, by Proposition 6.3
\[
\sum_{i=1}^{v} v_i / \sum_{i=1}^{v} u_i t_i \approx 1.
\]

But the denominator here is finite. Therefore
\[
\sum_{i=1}^{v} v_i \approx \sum_{i=1}^{v} u_i t_i,
\]
and, thus
\[
z_m = \left| \sum_{i=1}^{v} v_i - \sum_{i=1}^{v} u_i t_i \right| \approx 0,
\]
is true for every finite \( m \). We have, then, \( z_m \leq 1/m \), for every finite \( m \). Hence, by overflow, we can find an \( \eta \approx \infty \) such that \( 0 \leq z_\eta \leq 1/\eta \approx 0 \) and hence
\[
\sum_{i=1}^{v} v_i \approx \sum_{i=1}^{v} u_i t_i.
\]

We have \( 0 \leq u_i < 1/\eta \) implies that \( u_i \approx 0 \). Hence, by Lemma 7.1,
\[
\sum_{i=1}^{v} u_i t_i \approx 0 \quad \text{and} \quad \sum_{i=1}^{v} v_i \approx 0.
\]
Thus, we have

\[
\sum_{i=1}^{v} v_i = \sum_{i=1}^{v} u_i + \sum_{0 < u < 1/\eta}^{v} v_i \approx \sum_{i=1}^{v} u_i + \sum_{u \geq 1/\eta}^{v} u_i \approx \sum_{u \geq 1/\eta}^{v} u_i .
\]

We also have the following useful lemma, which implies approximate versions of the fundamental theorems, which we give below.

**Lemma 7.3.** Let \( u_i \) and \( t_i \), for \( 1 \leq i \leq v \), be terms such that \( u_i \approx x \) and \( t_i > 0 \) for every \( i \) with \( 1 \leq i \leq v \). Then

\[
\sum_{i=1}^{v} t_i \approx x.
\]

**Proof.** We have \( u_i = x + v_i \), with \( v_i \approx 0 \) for \( i = 1, \ldots, v \). Then

\[
\sum_{i=1}^{v} u_i t_i = \sum_{i=1}^{v} (x + v_i) t_i = x \sum_{i=1}^{v} t_i + \sum_{i=1}^{v} v_i t_i .
\]

By Lemma 7.1,

\[
\sum_{i=1}^{v} v_i t_i \approx 0 .
\]

We already introduced in §2 the abbreviations for sums, \( \sum_{a}^{c} f[u, v] \). If \( u \) is a geometric subdivision of order \( v \) of \([d, b]\), \( d \leq a < c \leq b \), and \( v \) is a selector for \( f \) and \( u \) on \([a, b]\), we also write \( \mu_a = \min f_{dom}(v_k) \geq a \) and \( \mu_c = \min f_{dom}(v_k) \geq c \). If \( f_{dom} \) is increasing, we write \( dv_j = f_{dom}(v_{j+1}) - f_{dom}(v_j) \) for \( \mu_a < j < \mu_c \). For \( f_{dom} \) decreasing, we introduce similar notation.

The next results give approximate forms of the fundamental theorems of calculus in their versions for sums.

**Corollary 7.4.** Let \( f \) be a continuous finite function on the interval \([a, b] \supseteq [x, x + y]\) with \( y \approx 0 \), \( y > 0 \), and \( a, b \) finite; let \( u \) be a geometric subdivision of order \( v \approx \infty \) of \([a, b]\) such that \( du/y \approx 0 \), and let \( v \) be a selector for \( f \) and \( u \) on \([a, b]\). If \( z \in [x, x + y] \cap dom f \), then

\[
\frac{\sum_{x+y} f[u, v]}{y} \approx f(z),
\]

and

\[
\frac{\sum_{x+y} f[u, v] - \sum_{x} f[u, v]}{y} \approx f(z).
\]

**Proof.** The second conclusion is obtained as follows. Suppose that \( x \in [u_j, u_{j+1}] \) (as we have mentioned before, one can define this \( j \)). By the assumption \( du/y \approx 0 \),
it is clear that \( a < u_j \leq x \leq u_{j+1} < x + y \). We then have

\[
\sum_{a}^{x+y} f[u,v] - \sum_{x}^{x+y} f[u,v] = \sum_{x}^{x+y} f[u,v] + f(t_j) \, du.
\]

Let \( p = \min_{u_{p+1} \geq x+y} \). We have

\[
\sum_{i=0}^{y} \, du_i + (u_j + 1) - x) + (x + y - u_p) = y.
\]

Hence, by Lemma 7.3,

\[
\frac{\sum_{i=0}^{y} \, f(t_i) \, du_i + f(t_j)(u_{j+1} - x) + f(z)(x + y - u_p)}{y} \approx f(z).
\]

Now

\[
\begin{align*}
\sum_{x}^{x+y} f[u,v] + f(t_j)du \\
= & \sum_{i=0}^{y} f(t_i)du + f(t_j)(u_{j+1} - x) + f(z)(x + y - u_p) \\
& + \frac{f(t_j)(x - u_j)}{y} - \frac{f(z)(x + y - u_p)}{y}.
\end{align*}
\]

Since \( du/y \approx 0 \) and \( f \) is finite,

\[
\frac{f(t_j)(x - u_j)}{y} \approx 0 \approx \frac{f(z)(x + y - u_p)}{y}.
\]

The first conclusion is obtained similarly.

**COROLLARY 7.5.** Let \( f \) and \( F \) be functions defined on the finite interval \([a,b]\) and such that \( \text{dom} f = \text{dom} F \). Let \( u \) be a geometric subdivision of \([a,b]\), and \( v \) a selector for \( f \) and \( u \) on \([a,b]\). Assume that \( f \) is finite and continuous on \([a,b]\), and that \( dF(x,y)/y \approx f(x) \) for all \( x, x + y \in [a,b] \cap \text{dom} f, y \approx \infty \). Then

\[
\sum_{a}^{b} f[u,v] \approx F(b') - F(a'),
\]

where \( a', b' \in \text{dom} f, a' \approx a \) and \( b' \approx b \).

**PROOF.** We have

\[
\sum_{a}^{b} f[u,v] = \sum_{i=1}^{v} f(t_i)du.
\]

Let \( f_{\text{dom}} \) be increasing (the proof for \( f_{\text{dom}} \) decreasing is similar). By Theorem
7.2 we have
\[ \sum_{a}^{b} f[u, v] \approx \sum_{\mu_a < j, j+1 < \mu_b} f(f_{\text{dom}}(v_j)) dv_j \]
\[ \approx \sum_{\mu_a \leq j, j+1 < \mu_b} dF(f_{\text{dom}}(v_j), dv_j). \]

By internal induction one can prove that for every \( m, \mu_a \leq m \leq n, \)
\[ \sum_{t=\mu_a}^{m-1} dF(f_{\text{dom}}(v_j), dv_j) = F(f_{\text{dom}}(v_m)) - F(f_{\text{dom}}(v_{\mu_a})). \]

Thus
\[ F(f_{\text{dom}}(v_{\mu_b})) - F(f_{\text{dom}}(v_{\mu_a})) \approx \sum_{a}^{b} f[u, v]. \]

§8. Definite integrals. We already introduced in §2 the definite integral in Axiom 18. As usual, we assume that all functions, unless explicitly expected, do not contain min. We now prove that if \( f \) is a continuous finite function on \([a, b]\) and \( b \) and \( a \) are finite, then the definition of the integral gives what we need:

**THEOREM 8.1.** If \( f \) is continuous and finite on \([a, b]\), \( u \) is a geometric subdivision of \([a, c]\) such that \( \frac{du}{c-b} \approx 0 \), and \( v \) is a selector for \( f \) and \( u \) on \([a, c]\), then

\[ \frac{\int_{a}^{b} f + \int_{b}^{c} f}{c-b} \approx \frac{\int_{a}^{c} f}{c-b}, \]

and

\[ \frac{\int_{b}^{c} f}{c-b} \approx \frac{\sum_{b}^{c} f[u, v]}{c-b}. \]

**PROOF.** We must prove that if \( u \) and \( u' \) are geometric subdivisions of \([a, c]\) and \( v \) and \( v' \) are selectors for \( f \) and \( u \) and \( u' \), respectively, on \([a, c]\), then

\[ \frac{\sum_{a}^{b} f[u, v] + \sum_{b}^{c} f[u, v]}{c-b} \approx \frac{\sum_{a}^{c} f[u, v]}{c-b}, \]

and

\[ \frac{\sum_{b}^{c} f[u, v]}{c-b} \approx \frac{\sum_{b}^{c} f[u', v']}{c-b}. \]

If \( c - b \approx 0 \), then the result is immediately obtained from Corollary 7.4. So assume that \( c \gg b \). We must prove

\[ \frac{\sum_{a}^{b} f[u, v] + \sum_{b}^{c} f[u, v]}{c-b} \approx \frac{\sum_{a}^{c} f[u, v]}{c-b}, \]

and

\[ \frac{\sum_{b}^{c} f[u, v]}{c-b} \approx \frac{\sum_{b}^{c} f[u', v']}{c-b}. \]
Since $c - b \neq 0$, from these results we obtain the premises (1) and (2) of Axiom 18, for integrals. The result (1) is easy to see, so we prove (2). We may assume that $u$ is a geometric subdivision of order $v$ of $[c, d]$ and $u'$, of order $\nu$ of the same $[c, d]$, and let $v$ and $v'$ be the corresponding selectors. (If the intervals for $u$ and $u'$ are not the same, we may change them to make them equal.) Let $u''$ be the geometric subdivision of $[c, d]$ of order $v \mu$, and $v''$ its selector. It is clear that $du''/du \approx 0$. Let $t$, $t'$, and $t''$ be the numbers selected according to the definition on page 129. Let

$$
\sum_{i} f = \sum_{u_i} f[u'', v''] + f(t'')du'' + f(t''_{v(i+1)-1})du''.
$$

Then, by Corollary 7.4,

$$f(t_i) \approx \sum_{i} f/du_i,$$

for $b \leq u_i < c$. Then, by Theorem 7.2

$$
\sum_{b} f[u'', v''] = \sum_{b} \left( \sum_{i} f \right) [u, v] \approx \sum_{b} f[u, v].
$$

Similarly, we prove that

$$
\sum_{b} f[u'', v''] \approx \sum_{b} f[u', v'],
$$

and, hence

$$
\sum_{b} f[u, v] \approx \sum_{b} f[u', v'],
$$

which is what was to be proved.

We can prove the usual theorems on integrals in an approximate form.

**Corollary 8.2.** Let $f$ be continuous on the finite interval $I$ and let $a$, $b$, $c \in I$, $a < c < b$. Then

$$
\int_{a}^{b} f \approx \int_{a}^{c} f + \int_{c}^{b} f.
$$

This is an immediate corollary of Theorem 8.1 (1).

**Theorem 8.3 (Fundamental Theorem 1).** Let $f$ be a continuous function on the finite interval $I$ and $a \in I$. Assume that $y \approx 0$, $y \neq 0$, and $x, x + y \in I$. If $z \in \text{dom} f$ with $z$ in the closed interval between $x$ and $x + y$, then

$$
\frac{\int_{x}^{x+y} f - \int_{x}^{x+y} f}{y} \approx f(z).
$$

**Proof.** Assume that $y \approx 0$, $y \neq 0$, $x, x + y \in I$, and $y > 0$. The case $y < 0$ is done similarly. Let $u$ be a geometric subdivision of $[a, b] \supset [x, x + y]$ and $v$ a
selector for \( f \) and \( u \) on \([a, b]\). Then, by Theorem 8.2,
\[
\int_a^{x+y} f - \int_a^x f \approx \sum_{x}^{x+y} f[u, v] / y,
\]
and, by Corollary 7.4,
\[
\sum_{x}^{x+y} f[u, v] / y \approx f(z).
\]

**THEOREM 8.4** (Fundamental Theorem 2). Let \( f \) and \( F \) be functions on the finite interval \([a, b]\) with the same domain, and let \( f \) be continuous. Suppose that \( dF(x, y)/y \approx f(x) \) for all \( y \neq 0 \), for \( x, x + y \in [a, b] \cap \text{dom } f \). Then
\[
\int_a^b f \approx F(b') - F(a'),
\]
where \( b' \approx b \) and \( a' \approx a \).

**PROOF.** Suppose that \( a \) and \( b \) are finite, and let \( u \) be a geometric subdivision of \([a, b]\) and \( v \) the corresponding selector. Then, by Theorems 8.1 and 7.5, since \( b - a \) is noninfinitesimal,
\[
\int_a^b f \approx \sum_{a}^{b} f[u, v] \approx F(b') - F(a').
\]

§9. Series and Transcendental Functions. We first introduce convergence of sequences. Just as previously, \( u \) stands for a term with a variable for natural numbers. If \( i \) is a natural number, \( u_i \) is a term with the variable replaced by \( i \). We say that \( u \) \( v \)-converges to \( x \) if for every \( \mu \leq v \), \( \mu \approx \infty \), we have \( u_{\mu} \approx x \), and that \( u \) \( v \)-converges if there is some \( x \) to which it \( v \)-converges. We say that the sequence \( u \) converges if it \( v \)-converges for every infinite \( v \).

A series is just a sequence \( v \) such that \( v_{\infty} = \sum_{i=1}^{\infty} u_i \). Then the series converges when it converges as a sequence. We have the following easy proposition:

**Proposition 9.1.** The series \( \sum_{i=1}^{\infty} u_i \) \( v \)-converges if and only if the tails \( \sum_{i=\mu}^{\infty} u_i \) are infinitesimal for all infinite \( \mu \leq v \), and it converges if and only if the tails are infinitesimal for every infinite \( \mu \).

From this proposition, we derive the comparison test:

**Theorem 9.2** (Comparison test). If \( v \approx \infty \) and for every \( n \approx \infty \), \( n \leq v \), we have \( |u_n| \leq |v_n| \), and the series \( \sum_{i=1}^{v} |v_i| \) \( v \)-converges, then the series \( \sum_{i=1}^{\infty} |u_i| \) also \( v \)-converges.

**Proof.** We have, by Proposition 9.1, that \( \sum_{i=\mu}^{v} |v_i| \approx 0 \) for every infinite \( \mu \leq v \). Then
\[
0 \leq \sum_{i=\mu}^{v} |u_i| \leq \sum_{i=\mu}^{v} |v_i| \approx 0
\]
for every infinite \( \mu \) (or \( \mu \leq v \)). The result is obtained from Proposition 9.1. \( \Box \)

We also have:
**Theorem 9.3 (Nelson [8]).** If $v \approx \infty$, $|u_i| \ll \infty$ for every $i \ll \infty$, and $\sum_{i=1}^{n} |u_i|$ $v$-converges, then $\sum_{i=1}^{v} |u_i| \ll \infty$, and hence the series $\sum_{i=1}^{n} |u_i|$ $v$-converges to a finite number.

**Proof.** We have that $\sum_{i=1}^{v} |u_i| \leq 1$ is true for all infinite numbers $n$. Then, by undertow, Theorem 6.1, the minimum $m$ must be finite. But

$$\sum_{i=1}^{m-1} |u_i| \leq (m-1) \max_{k} |u_k|.$$ 

Hence the sum

$$\sum_{i=1}^{v} |u_i| = \sum_{i=1}^{m-1} |u_i| + \sum_{i=m}^{v} |u_i|$$

is finite.

We assume that we have proved by internal induction (which is easily done) that

$$x > 0 \rightarrow (y > x \rightarrow y^n \geq x^n).$$

We also can give an easy internal inductive proof of the inequality, for $s > 0$,

$$(1 + s)^v \geq 1 + vs.$$ 

Also, if $s$ is noninfinitesimal, then $1/s$ is finite. Hence, if $v$ is infinite, then $1/v \approx 0$ and $1/(vs) \approx 0$. Thus, $vs$ is infinite.

Suppose, now, that $r \gg 1$ and $v \approx \infty$. We have that $r = 1 + s$, with $s \gg 0$. Therefore

$$r^v \geq (1 + s)^v \geq 1 + vs \approx \infty,$$

i.e., $r^v \approx \infty$. Hence, we also have that if $r \ll 1$ and $v \approx \infty$, then $r^v \approx 0$.

We prove by internal induction, as usual, that for $r > 0$ and $a > 0$

$$\sum_{i=0}^{n} ar^i = a \frac{1 - r^{n+1}}{1 - r}.$$ 

Then, if $0 < r \ll 1$ and $a$ is finite, the series $\sum_{i=0}^{n} ar^i$ converges to $a/(1 - r)$, which is also finite. Thus, we prove the ratio test:

**Theorem 9.4 (Ratio test).** If $u_i \geq 0$ is finite for every finite $i$, and there is an $r$, $0 < r \ll 1$, such that $u_{\mu+1}/u_\mu \approx r$ for every infinite $\mu \leq v$, then the series $\sum_{i=1}^{n} u_i$ $v$-converges to a finite number.

**Proof.** Let $\mu \approx \infty$, $\mu \leq v$. Then $u_{\mu+1}/u_\mu \approx r \leq r + \frac{1}{n} \ll 1$, where $n$ is a finite number.

Let $s = 1 + \frac{1}{n}$. Then we have $u_{\mu+1} \leq su_\mu$ for every infinite $\mu \leq v$. Hence, by undertow, Theorem 6.1, the minimum number $m$ such that $u_{p+1} \leq su_p$ for every $m \leq p \leq v$ is finite. By induction we prove that $u_{m+p} \leq u_ms^p$ for every $p \leq v - m$. Since $u_m$ is finite, the geometric series $\sum_{p=0}^{n} u_ms^p (v - m)$-converges to a finite
number. Then, by the comparison test, \( \sum_{p=0}^{n} u_{m+p} \) also \((v - m)\)-converges to a finite number, and so \( \sum_{i=1}^{n} u_{i} \) \(v\)-converges to a finite number. \(\square\)

We need a few theorems about series of functions, in our case of terms with a variable, say \(x\). The following are not in every case the best theorems that can be proved, but they are enough for our purposes.

**THEOREM 9.5.** Suppose that the series \( \sum_{i=1}^{n} u_{i}(x) \) \(v\)-converges, \( v \approx \infty \), for every \( x \) in a finite interval \( I \), and that \( u_{i}(x) \) is continuous on \( I \) for every \( i \), \( 1 \leq i \leq v \). Then \( \sum_{i=1}^{\mu} u_{i}(x) \) is continuous on \( I \), for every \( \mu \leq v \).

**PROOF.** Let \( x, y \in I \), \( x \approx y \). For every finite \( n \), we proved by external induction that

\[
\left| \sum_{i=0}^{n} u_{i}(x) - \sum_{i=0}^{n} u_{i}(y) \right| \leq \sum_{i=0}^{n} \left| u_{i}(x) - u_{i}(y) \right| \leq \frac{1}{n}
\]

By overflow, Theorem 6.2, there is an infinite \( \eta \) such that for all infinite \( \mu \leq \eta \)

\[
\left| \sum_{i=0}^{\mu} u_{i}(x) - \sum_{i=0}^{\mu} u_{i}(y) \right| \leq \frac{1}{\mu} \approx 0.
\]

If \( \eta \geq v \geq \mu \), we are done. Assume that \( \eta < \mu \leq v \). Then

\[
\left| \sum_{i=0}^{\mu} u_{i}(x) - \sum_{i=0}^{\mu} u_{i}(y) \right| \leq \left| \sum_{i=0}^{\eta} u_{i}(x) - \sum_{i=0}^{\eta} u_{i}(y) \right| + \left| \sum_{i=\eta+1}^{\mu} u_{i}(x) - \sum_{i=\eta+1}^{\mu} u_{i}(y) \right|
\]

\[
\leq \left| \sum_{i=0}^{\eta} u_{i}(x) - \sum_{i=0}^{\eta} u_{i}(y) \right| + \left| \sum_{i=\eta+1}^{\mu} u_{i}(x) \right| + \left| \sum_{i=\eta+1}^{\mu} u_{i}(y) \right|
\]

\(\approx 0.\) \(\square\)

We assume that the definite integral has been extended to a lower limit \(a\) and an upper limit \(b\), both finite, with \(a \geq b\), as it is usually done.

For development of the transcendental functions, although we shall not use it here, we also need a theorem about power series, whose proof is in [12]:

**THEOREM 9.6.** Let \( I = (\alpha, \beta) \), where \( \beta \) is finite, and suppose that the series \( \sum_{i=1}^{n} |a_{i}| x^{i} \) \(v\)-converges on \( I \), \( v \approx \infty \). Then:

(1) The series \( \sum_{i=1}^{n} a_{i} x^{i-1} \) and \( \sum_{i=1}^{n} \left( a_{i} / (i + 1) \right) x^{i} \) \(v\)-converge on any \(|x| \ll \beta|\).  
(2) The series \( \sum_{i=1}^{n} a_{i} x^{i} \) is differentiable, and

\[
\left( \sum_{i=1}^{n} a_{i} x^{i} \right)'(x) \approx \sum_{i=1}^{n} a_{i} x^{i-1},
\]

for every \( n \leq v \) and any \(|x| \ll \beta|\).

(3) \(\int_{\alpha}^{\beta} \sum_{i=1}^{n} a_{i} x^{i} \approx \sum_{i=1}^{n} \frac{a_{i}}{i+1} x^{i+1},\)

for every \( n \leq v \) and any \(|a| \ll \beta|\).
We can now define the main transcendental functions. We first introduce the natural logarithm by the following axiom:

$$\log x = \int_1^x \frac{1}{t} \, dt.$$  

We consider \(\log\) as a function, take \(\log_{\text{dom}}(x) = I(x)\), the identity function, and \(\varphi_{\log}(x) \to 0 \ll x \ll \infty\). In the usual way, we can obtain the main properties of the logarithm.

By Theorem 8.3, \(\log'(x) \approx 1/x\), for every \(x\) with \(\infty \gg x \gg 0\). Then, if we take any interval \([a, b]\), with \(0 \ll a < b \ll \infty\), \(\log'\) is bounded on the interval by a finite \(M\). Thus, by Theorem 4.2, \(\log\) is Lipschitz continuous on \([a, b]\). Also, \(1/x \gg 0\) for \(x \gg 0\). Thus, by Theorem 5.7, \(\log\) is strictly increasing on all its domain. Therefore, \(\log\) can be \(f_{\text{dom}}\) for a function \(f\). We then define an "almost" exponential function:

$$\text{aexp}(x) = y \mapsto x = \log y,$$

with \(\text{aexp}_{\text{dom}} = \log\) and \(\varphi_{\text{aexp}}(x) \to 0 \ll x \ll \infty\). Using Theorem 5.7, we get that \(\text{aexp}\) is differentiable on its domain, and we can calculate its derivative, \(\text{aexp}' \approx \text{aexp}\). Thus \(\text{aexp}\) is increasing.

In a similar way, we define the arctan:

$$\arctan x = \int_0^x \frac{1}{1 + t^2} \, dt.$$  

As above, we consider \(\arctan\) as a function, with \(\arctan_{\text{dom}}(x) = I(x)\) and \(\varphi_{\arctan}(x) \to |x| \ll \infty\). We take \(\pi = 4 \arctan 1\). As for the logarithm, by Theorem 8.3, \(\arctan'(x) \approx 1/(1 + x^2)\) for any finite \(x\). So that \(\arctan\) can be \(f_{\text{dom}}\). Thus, we define the inverse, the almost tangent:

$$y = \text{atan} x \mapsto x = \arctan y.$$  

We extend this function periodically by taking \(\text{atan}(x + n\pi) = \arctan x\), where \(n\) is any integer. The domain \(\text{atan}_{\text{dom}} = \arctan\) and \(\varphi_{\text{atan}}(x) \to |x| \ll \infty\).

The definitions of the inverse functions (almost exponential and tangent) are justified by Theorem 5.7. We must use the same theorem for obtaining the derivatives. With the definitions introduced here, the proofs of the approximate form of the algebraic properties of these functions are the usual ones.

We cannot prove, however, the inverse functions, i.e., the almost exponential and tangent, have the right domains, i.e., all finite numbers for the almost exponential and the finite numbers different from \((2n + 1)\pi/2\) for the almost tangent. The most one can do, for the almost exponential for instance, is to prove that for any finite number \(x\) there is a \(y \approx x\) in its domain (see the remark after Theorem 5.7), which is probably sufficient for most uses in theoretical physics. In order to obtain functions defined everywhere, we use Taylor series approximations.

As an example, we take the series for \(\text{aexp}\). We observe that, by the ratio test, if \(x\) is finite, the series \(\sum_{i=0}^{\infty} x^i / i!\) converges. Since \(\log 1 \approx 0\), we have that if \(x \approx 0\), \(x \in \text{dom aexp}\), \(\text{aexp}(x) \approx 1\). We then show, by external induction, that if
$x \approx n + 1$ and $y \approx n$, where $n$ is a finite natural number and $x, y \in \text{dom} aexp$, then $aexp(x) \approx aexp(y) aexp(1)$, and $aexp(x)$ is finite.

We first prove, using Theorem 5.7, that, for every finite $x$ and $dx \approx 0$ such that both $x$ and $x + dx$ are in the domain of $aexp$,

$$\frac{aexp(x + dx) - aexp(x)}{dx} \approx aexp(x).$$

Thus, we can take $aexp' = aexp$. By a recursive definition, for finite $n$ we have $aexp^{(n)} = aexp$, and we prove, by external induction, that for every finite $n$

$$\frac{aexp^{(n)}(x + dx) - aexp^{(n)}(x)}{dx} \approx aexp^{(n+1)}(x).$$

We then have $aexp^{(n)}(0) \approx 1$. By Taylor’s Theorem 5.8, for every finite $n$

$$\left| aexp(x) - \sum_{i=0}^{n} \frac{x^i}{i!} \right| \leq \frac{aexp(|x|)|x|^{n+1}}{(n+1)!} + \frac{1}{n}.$$

By overflow, Theorem 6.2, the same is true for every $\mu \leq \nu$, for a certain infinite $\nu$.

Let $\mu \approx \infty$, and let $m$ be a finite number such that $m > 2|x|$. Hence $|x|/n < 1/2$ for every $n \geq m$. We have

$$\frac{|x|^\mu}{\mu!} = \frac{|x|^m}{m!} \frac{|x|}{m+1} \frac{|x|}{m+2} \ldots \frac{|x|}{\mu} \leq \frac{x^m}{m!} \frac{1}{2^{\mu-m}} \leq \frac{|x|^m}{2^\mu} \approx 0.$$

Therefore, we have proved that, for every infinite $\mu \leq \nu$,

$$aexp(x) \approx \sum_{i=0}^{\mu} \frac{x^i}{i!}.$$

Since the series $\sum_{i=0}^{\mu} x^i/i!$ is convergent, we have that this is true for every infinite $\mu$.

We now define the exponential function:

$$\exp x = e^x = \sum_{i=0}^{\nu} \frac{x^i}{i!},$$

for all finite $x$.

Let $x$ be finite. Then there is a $y \approx x$ such that $y \in \text{dom} aexp$. Thus, by Theorem 9.5,

$$e^x = \sum_{i=0}^{\nu} \frac{x^i}{i!} \approx \sum_{i=0}^{\nu} \frac{y^i}{i!} \approx aexp(y).$$

By the definition of $aexp$ we conclude that $aexp(y) = z \leftrightarrow y = \log z$. Hence, for any $z_1$ such that $e^x \approx z_1$, we have that there is $y \approx x$ such that $aexp(y) \approx z_1$. Let $aexp(y) = z$. Then $z \approx z_1$ and $y = \log z \approx \log z_1$. Thus, $x \approx \log z_1$. On the other hand, if $x \approx \log z_1$ and $y = \log z_1$, then by the Taylor approximation of $aexp$ we have $z_1 = aexp(y) \approx e^x$. Hence, we have proved that $e^x \approx z_1 \leftrightarrow x \approx \log z_1$. 


In the case of the trigonometric functions, it seems simpler to define first an almost sine and an almost cosine by the formulas:

\[
\begin{align*}
\text{asin } x &= \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \\
\text{acos } x &= \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}},
\end{align*}
\]

with their domains the same as the domain of the \( \text{atan} \). We then use Taylor series to give a definition of sin and cos on all finite numbers. The procedure is similar to that described above for the exponential function.

REFERENCES


