LETTER

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A Spline Framework for Estimating the EEG Surface Laplacian Using the Euclidean Metric

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This letter develops a framework for EEG analysis and similar applications based on polyharmonic splines. This development overcomes a basic problem with the method of splines in the Euclidean setting: that it does not work on low-degree algebraic surfaces such as spherical and ellipsoidal scalp models. The method's capability is illustrated through simulations on the three-sphere model and using empirical data.

1 Introduction

The surface Laplacian model of EEG data has been demonstrated to be a valuable tool in EEG analysis. Basically the surface Laplacian estimates the radial current density entering or leaving the scalp beneath electrode sites (Hjorth, 1975; Perrin, Bertrand, & Pernier, 1987; Perrin, Pernier, Bertrand, Giard, & Echallier, 1987; Perrin, Pernier, Bertrand, & Echallier, 1989, 1990; Gevins, Leong, Smith, Le, & Du, 1995; Nunez & Srinivasan, 2006). This quantity is more robust against volume conduction effects of the conductive head than conventional scalp potentials, which in general provide only a blurred copy of the cortical potential distribution. Importantly, this improvement in spatial resolution is achieved without making any unsupported assumption about brain generators, such as the number of dipoles and their locations in the brain. Furthermore, the surface Laplacian trivially solves the electrode reference problem that merits serious attention in EEG studies. It also has the advantage of being relatively robust against artifacts generated in regions not covered by the electrode cap, such as the potentials due to eye movements. These properties provide a substantial motivation for using the surface Laplacian in the spatial analysis of EEG signals.

However, the benefits of the surface Laplacian might not be fully achieved in practice due to estimation errors. By definition, the surface Laplacian is a continuous mathematical operator that can be applied only approximately to empirical data (Jackson, 1999). The classic scheme to estimate the surface Laplacian is the method due to Hjorth (1975), which is essentially a second-order, finite difference approach that uses the potential at a given electrode and its four nearest neighbors to make a local estimate. Due to its mathematical simplicity, Hjorth’s method offers a good framework for theoretical explorations. However, it assumes an unrealistic piecewise planar scalp and does not account suitably for unequal interelectrode separations or for estimations at border electrode sites. Furthermore, the method is not endowed with a mechanism for handling noise in the potential distribution.

Perrin et al. (1989) introduced a method that uses a global interpolant rather than finite differences to perform the Laplacian differentiation. This method is commonly referred to as the spherical Laplacian because it is based on a spherical pseudo-spline approach developed by Wahba (1981, 1990). Since splines fall into the wide class of mesh-free methods (Fasshauer, 2007), the spherical Laplacian has the advantage of working for arbitrary interelectrode distances. Moreover, it provides continuous predictions of the Laplacian distribution all over the head. Despite some shortcomings, the spherical Laplacian has proved to be valuable in practice, and it is probably the most widely used Laplacian technique.

Splines arose from problems in applied mathematics (Schoenberg, 1946) and physics (Harder & Desmarais, 1972; Green & Silverman, 1994). After the mathematical work of Duchon (1977), the method has become an indispensable tool in multivariate data analysis. The approach developed by Wahba (1981, 1990) is the earliest work we are aware of to address a serious problem with the spline framework: that the method does not apply to data points on spheres. This problem occurs because the surface’s equation constrains the interpolation conditions, making them insufficient to eliminate all degrees of freedom of the interpolation problem. Wahba’s solution circumvents this problem by replacing the Euclidean distance by geodesic distance.

Unfortunately, the method of using geodesic distance does not extend easily to other quadratic geometries in which splines also do not work. An example is the case of data points on ellipsoids, which is useful in EEG studies. Moreover, the geodesic distance is totally inappropriate to use in Euclidean spaces, and so does not account for most of the realistic head shapes.

The major contribution of this letter is the development of a method that extends splines to non-Euclidean scalp models without using the geodesic norm. Rather, we employ QR factorization, which decomposes a data matrix into an orthogonal, $Q$, and an upper triangular, $R$, matrix, and minimum norm solution to handle the singularity problem in the interpolation
conditions. The usefulness of this framework goes beyond that of the spherical Laplacian, since it accounts for Laplacian estimations and smooth reconstruction of EEG data in arbitrary scalp shapes. Importantly, the required mathematical operations are given in a closed form and consist basically of a linear transformation of the scalp potential distribution. This speeds up computation and makes it easy to integrate the algorithm with other linear analysis methods. The special case in which the scalp is represented by a perfect sphere is developed in detail and evaluated by means of simulations. The performance of the method on empirical data is also studied in the context of a two-class discrimination task.

2 Methods

2.1 Spline Interpolation. Using splines to interpolate or smooth data amounts to solving a linear system of the following form (Duchon, 1977; Meinguet, 1979; Wahba, 1990; Green & Silverman, 1994; Eubank & Eubank, 1999):

\[
\begin{pmatrix}
  (K + N\lambda I) & T \\
  T^T & 0
\end{pmatrix}
\begin{pmatrix}
  c \\
  d
\end{pmatrix}
= \begin{pmatrix}
  v \\
  0
\end{pmatrix}.
\]

The unknowns \( c = (c_1, \ldots, c_N)' \) and \( d = (d_1, \ldots, d_M)' \) are the coefficients that expand the spline interpolant defined below. The vector \( v = (v_1, \ldots, v_N)' \) represents the instantaneous potential distribution sampled at electrode locations \( r_1, \ldots, r_N \in \mathbb{R}^3 \). The matrices \( K \) and \( T \) have dimensions \( N \times N \) and \( N \times M \), respectively, and \( (K)_{ij} = \|r_i - r_j\|^{2m-3} \) and \( (T)_{ij} = \phi_j(r_i) \). The parameter \( m \) assumes only integer values and is required to satisfy \( 2m > 3 \). The functions \( \phi_1, \ldots, \phi_M \) are all monomials in three variables of degree less than \( m \). There are \( M = \binom{m+2}{3} \) such monomials, and we express them in the form

\[
\phi_\ell(r) = x^i y^j z^k,
\]

where \( \ell = i + j + k + 1, 0 \leq i \leq m - 1, 0 \leq j \leq i, \) and \( 0 \leq k \leq j \). For instance, for \( m = 2 \), we have that \( M = 4 \) and \( \phi_1(r) = 1, \phi_2(r) = x, \phi_3(r) = y, \) and \( \phi_4(r) = z \). The parameter \( m \) must be such that \( N > M \), that is, \( T \) must have more rows than columns. Otherwise problem 2.1 is ill posed.

The parameter \( \lambda \geq 0 \) provides a useful mechanism to reduce spatial distortions in the measurements due to noise (Wahba, 1990). Adjusting \( \lambda \) regulates the trade-off between minimizing the squared error of the data fitting and smoothness. The input data set \( v \) is fully recovered for \( \lambda = 0 \), while a waveform that ignores all of the features of the data is obtained for large \( \lambda \). Between these extremes lies an “optimal” \( \lambda \)-value whose estimation requires a special technique, such as the method of cross-validation and generalized cross-validation (Golub, Heath, & Wahba, 1979; Craven &
Smooth splines determine an interpolant, \( v_\lambda(r) \), for the given data set as the unique minimizer of the penalized sum of squares (Wahba, 1990; Green & Silverman, 1994),

\[
F(v) = \frac{1}{N} \sum_{i=1}^{N} [v_i - v(r_i)]^2 + \lambda J_m(v),
\]

where \( J_m(v) \) is a measure of the roughness of \( v \) (Wahba, 1990; Green & Silverman, 1994; Eubank & Eubank, 1999). The minimizer of \( F(v) \) has the general form

\[
v_\lambda(r) = \sum_{i=1}^{N} c_i \|r - r_i\|^{2m-3} + \sum_{t=1}^{M} d_t \phi_t(r),
\]

where the \( \lambda \)-dependency of \( v_\lambda \) is embedded in the parameters \( c_i \)'s and \( d_t \)'s.

Once system 2.1 is solved, the potential distribution can be estimated at any scalp location by equation 2.4. If \( \lambda \neq 0 \), then this formula smoothly reconstructs the original potential distribution in the spatial domain.

### 2.2 The Problem with Splines on Spherical and Ellipsoidal Scalps.

An important remark about the above approach is that it does not account for surface Laplacian estimations on spherical or ellipsoidal scalp models. This is because only if \( m > 2 \) is system 2.1 nonsingular for sample points on a quadratic surface. As the interpolant possesses discontinuous second derivatives at all electrode coordinates for \( m = 2 \), this case is not suitable for computing the surface Laplacian (see equation B.11 in appendix B and equations C.17 in appendix C).

Without loss of generality, let us consider the \( m = 3 \) case for which the interpolant has continuous derivatives up to the second order, as required for surface Laplacian estimations. In this case, \( T \) is the \( N \times 10 \) matrix of the form\(^1\)

\[
T = \begin{pmatrix}
1 & x_1 & y_1 & z_1 & x_1^2 & x_1 y_1 & x_1 z_1 & y_1^2 & y_1 z_1 & z_1^2 \\
1 & x_N & y_N & z_N & x_N^2 & x_N y_N & x_N z_N & y_N^2 & y_N z_N & z_N^2 
\end{pmatrix}
\]

\(^1\)Notice that the columns of \( T \) follow the order in which the monomials \( \phi_i \) appear in equation 2.2.
If the sampling points are a subset of a sphere or ellipsoid, then those columns of $T$ spanned by the monomials $1, x^2, y^2,$ and $z^2$ (i.e., columns 1, 5, 8, and 10 of $T$) are linearly dependent due to the constraint

$$\frac{x_i^2}{a^2} + \frac{y_i^2}{b^2} + \frac{z_i^2}{c^2} = 1, \quad i = 1, \ldots, N. \quad (2.6)$$

Here $a, b, c > 0$ represent the semi-axes of the ellipsoid ($a = b = c$ for the sphere). Therefore, $T$ is rank deficient, that is, $T$ is rank 9, and so equation 2.1 is singular for $m = 3$. The same holds for $m > 3$. The $m = 2$ case is an exception because no second- or higher-degree polynomial is involved in the construction of $T$.

### 2.3 The Spherical Laplacian

Wahba (1981) proposed a scheme for interpolating on spheres that consists of minimizing equation 2.3 using geodesic distance rather than the Euclidean distance. This procedure, which applies only to spherical models, yields the interpolant

$$v_i^2(r) = \sum_{i=1}^{N} c_i g_m(r, r_i) + d, \quad (2.7)$$

where

$$g_m(r, r_i) = \frac{1}{4\pi} \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{\ell m_s(\ell + 1)m_s} P_{\ell}(\hat{r} \cdot \hat{r}_i), \quad m_s > 1. \quad (2.8)$$

Here, $P_{\ell}$ is the Legendre polynomial of degree $\ell$, and caret denotes unit vectors. The coefficients $c_i$'s and $d$ are still determined by equation 2.1, but now $(K)_{ij} = g_m(r, r_i)$ and the matrix $T$ is a vector of ones (i.e., $\dim(T) = N \times 1$). Wahba's approach became popular in EEG thanks to the work of Perrin et al. (1989), who were the first to use it to estimate the surface Laplacian on the spherical scalp model. Noteworthy also is the detailed study presented by Babiloni et al. (1995), in which errors due to the unrealistic spherical approximation, amount of smoothness, and the value of $m$ were statistically evaluated using variants of the 10/20 EEG system and noise of different magnitudes and spatial frequencies.

In spherical coordinates,

$$\Delta_{\text{surf}} f = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \varphi^2} \right], \quad (2.9)$$

where the polar angle $\theta$ is the angle down from the z-axis (vertex) and the azimuthal angle $\varphi$ is the angle around from the x-axis (nasion). As the Legendre polynomials satisfy the relation (Jackson, 1999, p. 110)

$$\Delta_{\text{surf}} P_{\ell}(\hat{r} \cdot \hat{r}_i) = -\frac{\ell(\ell + 1)}{r^2} P_{\ell}(\hat{r} \cdot \hat{r}_i), \quad (2.10)$$
we obtain from equation 2.7 that

$$\Delta_{\text{surf}} v^r_m(r) = -\frac{1}{r_{\text{head}}^2} \sum_{i=1}^{N} c_i g_{m, -1}(r, r_i), \quad m_s > 1,$$

(2.11)

where $r_{\text{head}}$ is the scalp radius. That is, the spherical Laplacian is approximately a spherical interpolant of order $m_s - 1$.

In the Euclidean setting, the parameter $m$ controls the degree of the interpolant, that is, the larger $m$ is, the higher is the degree of the interpolant, and the orthogonality condition $T'c = 0$, which is embedded in equation 2.1, bounds the interpolant outside the region covered by the electrode sites. For instance, for a planar scalp model and for $m = 2$, this condition cancels any second- or higher-degree term in the expansion of $v_1(r)$ for $r$ large, thus forcing the interpolant to approach a plane a long distance from the electrode sites (Harder & Desmarais, 1972; Perrin, Pernier, et al., 1987). In turn, as the spherical interpolant is defined in a compact space, it requires periodic rather than asymptotic conditions. Hence, no Legendre polynomial is eliminated from the interpolant by the orthogonality condition in the spherical setting.

Although the performance of the spherical Laplacian with empirical data is usually satisfactory, precaution needs to be taken to avoid numerical errors due to truncations in equation 2.8. More important, for large $\ell$, the term $(2\ell + 1)/(\ell^{m_s} (\ell + 1)^{m_s})$ in equation 2.8 approaches 0 rapidly. Thus, high-frequency spatial contributions to $g_{m_s}(r, r_i)$ fall off rapidly, and the interpolation acts as a spatial filter that weights down short wave-length features. The larger $m_s$ is, the more features are smoothed out. This behavior causes blurring distortions that may jeopardize the intended enhancement in spatial resolution. Also important is the fact that the condition number for the coefficient matrix of equation 2.1 is usually high, and it gets rapidly worse by increasing $m_s$ for montages with more than 64 electrodes. Therefore, one has little room for selecting $m_s$ in practice—typically $2 \leq m_s \leq 6$ (Babiloni et al., 1995).

### 2.4 The Minimum-Norm Approach

In this section, we use the pseudo-inverse matrix to circumvent the problem of using splines on quadratic surfaces in the Euclidean setting. The rank deficiency of the matrix $T$ on quadratic surfaces is a problem that affects only the unknowns $d$. In fact, consider the decomposition of $T$ using the QR factorization

$$T = (Q_1, Q_2) \begin{pmatrix} R & O \end{pmatrix},$$

(2.12)

where $Q_1 \in \mathbb{R}^{N \times M}$ and $Q_2 \in \mathbb{R}^{N \times (N-M)}$ are orthonormal and $R \in \mathbb{R}^{M \times M}$ is upper triangular. According to equations 2.1 and 2.12, we have that $T'c = 0$.
and \( T'Q_2 = 0 \), that is, the matrix \( T \) is orthogonal to both \( c \) and \( Q_2 \). This implies that \( c \) is a vector in the column space of \( Q_2 \). As \( Q_2 \) is well conditioned, these unknowns can be determined even if \( T \) is rank deficient. In fact, according to Wahba (1990, p. 13), we have that

\[
c = Q_2 \left[ Q_2' \left( K + N\lambda I \right) Q_2 \right]^{-1} Q_2' v,
\]

\[
Rd = Q_1' \left( v - Kc - N\lambda c \right).
\]  

The solution for the unknown \( d \)'s depends on the scalp geometry. If \( T \) is of full rank, then \( d \) is unique, and it can be determined either by inverting the (nonsingular) square matrix \( R \) or, more efficiently, by back-substitution. This is the case of most of the realistic scalp shapes (Law & Nunez, 1991).

The case of the widely used spherical scalp model, whose geometry greatly facilitates the Laplacian differentiation, requires attention. In order to handle this case and other quadratic surfaces as well, we adopt the minimum norm solution, which assumes \( \min \| d \| \) among all possible solutions for equation 2.13b. The minimum-norm solution of equation 2.13b is computed as

\[
d = R^+ Q_1' \left( v - Kc - N\lambda c \right),
\]

where \( R^+ \) denotes the Moore-Penrose pseudo-inverse of \( R \). The pseudo-inverse satisfies the conditions \( RR^+ R = R \) and \( R^+ RR^+ = R^+ \) and is usually defined by singular-value decomposition (Golub & Varah, 1996). Let \( R = U \Sigma V' \), where \( U \) and \( V \) are \( M \times M \) unitary matrices and \( \Sigma \) is an \( M \times M \) diagonal matrix containing the singular values of \( R \) in descending order. Then the pseudo-inverse admits the representation \( R^+ = V \Sigma^+ U' \), where \( \Sigma^+ \) is the matrix formed by replacing every nonzero element in \( \Sigma \) by its reciprocal. The solution, equation 2.14, is unique, and any other solution to equation 2.13b differs from it by a vector in the null space of \( R \).

### 2.5 The Surface Laplacian Linear Transformation.

The advantage of using QR factorization to solve the interpolation problem is twofold. First, it restricts the rank-deficiency problem of \( T \) within the space of solutions of equation 2.13b. Second, it allows us to formulate the problems of EEG smoothing and Laplacian differentiation in terms of linear transformations. For this, we introduce the data-independent matrices \( C_\lambda \) and \( D_\lambda \) of dimensions \( N \times N \) and \( M \times N \), respectively, which are such that \( c = C_\lambda v \) and \( d = D_\lambda v \)—that is,

\[
C_\lambda = Q_2 \left[ Q_2' \left( K + N\lambda I \right) Q_2 \right]^{-1} Q_2',
\]

\[
D_\lambda = R^+ Q_1' \left( I - KC_\lambda - N\lambda C_\lambda \right).
\]
Since \( R^+ \) coincides with \( R^{-1} \) when \( R \) is invertible, these expressions are general, being applicable to either Euclidean or non-Euclidean geometries.\(^2\)

The smooth reconstruction of EEG data is performed by linearly transforming the data, according to

\[
\mathbf{v}_\lambda(t) = \mathbf{S}_\lambda \mathbf{v}(t),
\]

where \( \mathbf{S}_\lambda \in \mathbb{R}^{N \times N} \) is the smoothing matrix

\[
\mathbf{S}_\lambda = \mathbf{K} \mathbf{C}_\lambda + \mathbf{T} \mathbf{D}_\lambda.
\tag{2.17}
\]

We have introduced the time variable to emphasize that \( \mathbf{S}_\lambda \) remains unchanged across time samples. The variable \( r \) was suppressed to stress that this formula holds only at the electrode sites. Since the computation of \( \mathbf{S}_\lambda \) does not involve differentiation, this matrix can be easily obtained for any scalp shape.

We observe that the coefficient matrices \( \mathbf{C}_\lambda \) and \( \mathbf{D}_\lambda \) are not affected by spatial differentiation. Hence, the surface Laplacian operator \( \Delta_{\text{surf}} \) can be represented, at the electrode sites, by the square matrix

\[
\mathbf{L}_\lambda = \mathbf{K} \mathbf{C}_\lambda + \mathbf{T} \mathbf{D}_\lambda.
\tag{2.18}
\]

Here \( \mathbf{K} \in \mathbb{R}^{N \times N} \) and \( \mathbf{T} \in \mathbb{R}^{N \times M} \) are the matrices of elements \((\mathbf{K})_{ij} = \Delta_{\text{surf}} \| \mathbf{r}_i - \mathbf{r}_j \|^{2n-3}\) and \((\mathbf{T})_{ij} = \Delta_{\text{surf}} \phi_i(\mathbf{r}_j)\), respectively.\(^3\) The Laplacian wave form is then given by

\[
\mathbf{w}_\lambda(t) = \mathbf{L}_\lambda \mathbf{v}(t).
\tag{2.19}
\]

Therefore, according to equations 2.16 and 2.19, EEG smoothing and surface Laplacian differentiation are operations that consist essentially of linearly transforming the given potential distribution. If the scalp geometry is set a priori (e.g., for a spherical scalp model) and provided the electrode configuration remains the same, the transforming matrices \( \mathbf{S}_\lambda \) and \( \mathbf{L}_\lambda \) are both subject independent and therefore need to be computed just once for a given \( \lambda \)-value.

As the operator \( \nabla_{\text{surf}}^2 \) vanishes at any monomial of degree less than 2, the first four columns of \( \mathbf{T} \) are always null. Hence \( \mathbf{T} \) and, consequently, \( \mathbf{L}_\lambda \) are inevitably rank deficient. This property may be relevant in some procedures. An example is the multiple-step approach of Carvalhaes, Perreau-Guimaraes, Grosenick, and Suppes (2009), which combines the surface Laplacian with independent component analysis (ICA) to improve single-trial classifications.

\(^2\)It is worth remarking that the pseudo-inverse is an expensive procedure to determine \( R^{-1} \).

\(^3\)Note that \( \mathbf{T} \) is a 0 matrix for the spherical Laplacian.
2.6 Simulation. Simulations based on the three-sphere volume conductor model were used to evaluate the performance of the surface Laplacian derived in the Euclidean setting. Brain, skull, and scalp were represented as concentric spheres of radii 8.0, 8.6, and 9.2 cm, respectively. The brain and scalp had the same resistivity of 300 $\Omega \times cm$, and the skull was 80 times more resistive. The brain sources were radial dipoles with polar ($\theta$) and azimuth ($\varphi$) angles in the ranges of 0 to 90 degrees and 0 to 360 degrees. Each dipole modeled a current of 1 $\mu$A flowing between poles separated by a constant distance of 1 mm. The current dipole is the basic element for representing neural activation in the EEG inverse problem (Braun, Kaiser, Kincses, & Elbert, 1997; Okada, Wu, & Kyuhou, 1997; Darvas, Pantazis, Kucukaltun-Yildirim, & Leahy, 2004). Although the surface Laplacian is not a three-dimensional source localization technique, the well-known connection between brain potential and scalp Laplacian distributions establishes the Laplacian as an important tool for source mapping. The objective of our simulations was therefore to assess the ability of the Euclidean approach to produce a reliable mapping of underlying cortical activity.

Appendix A presents the analytic solution to the forward problem used in our simulations. The potential and Laplacian distributions due to a single radial dipole is given by equations A.8. The more general case of linear combinations of radial dipoles is straightforwardly accounted for by linearly superposing the solutions for the dipoles alone. The formal expression for the Laplacian differentiation of equation 2.4 on a spherical scalp is presented in appendix B. The Matlab codes implementing these computations are available as supplementary material at http://www.mitpressjournals.org/doi/suppl/10.1162/NECO_a_000192.

The simulations were performed in the noiseless regime. The generated potentials were spatially sampled using geodesic partitions of 64, 128, and 256 sites (Tucker, 1993). These distributions referred to the value of the potential at the vertex—$\theta = 0$. As the surface Laplacian is reference free, the reference electrode was included in all Laplacian estimations, thereby augmenting the montage by one electrode. The Laplacian wave forms were reconstructed by applying the Euclidean Laplacian to the generated potential distributions. The parameter $m$ varied in the range from 3 to 6.$^4$

The root mean squared error (RMSE) was used to evaluate the accuracy of the fitting. The RMSE was defined as

$$E = \sqrt{\frac{\sum_{i=1}^{N} (w_i - w_{i}^{true})^2}{N}},$$

where $N$ is the number of samples (electrodes) and $w_i$ and $w_{i}^{true}$ are the $i$th values of the estimated and "true" Laplacians.

$^4$Larger $m$-values would violate the condition $N > M$ for the montage with 64 electrode sites.
2.7 Empirical Data. The performance of the Laplacian method was evaluated on empirical data within the context of a two-class imagination task. Eleven participants (S1–S11), ranging in age from 21 to 25 years, participated in an experiment in which they were randomly presented either a visual stimulus, consisting of a red stop sign flashed on a computer screen, or an auditory stimulus consisting of the English word *go*, uttered by a male native speaker of English, digitized at 22 kHz, and presented by computer speaker. Each presentation lasted 300 ms and was followed by a period of 700 ms during which the participant saw a blank screen. Immediately after this period, a fixation point consisting of a small white + sign was shown on the center of the screen for a period of 300 ms. During this time, the participant was instructed to form a vivid mental image of either the just-presented stimulus (S1–S7) or the alternative stimulus (S8–S11). Participant imagining was followed by another 700 ms of blank screen, after which the trial ended. Prior to recording the EEG signal, the participants were given the opportunity to practice with trial samples.

The EEG signal was recorded using a 64-channel Neuroscan system (Neurosoft, Sterling, U.S.A.) with monopolar electrodes referenced to linked mastoids. A pair of bipolar electrodes was placed at the outer canthi to record the horizontal electrooculogram (HEOG) and another pair on the sub- and supraorbital ridges of the left eye for vertical electrooculogram (VEOG). The 62 monopolar electrodes were placed at the scalp locations of the 5% system (Oostenveld & Praamstra, 2001), with the exceptions of the electrodes Nz, AF1, AF2, AF5, AF6, T9, T10, P9, P5, P6, P9, P10, PO, or I, which were not included. The signal was recorded continuously across stimulus presentation and imagination tasks.

Each session consisted of 30 blocks of 20 trials with regular breaks controlled by the participant by key press. The analog signal was digitized at a sample rate of 1 kHz after being bandpass-filtered at the bandwidth of 0.1 Hz to 300 Hz and through a 60 Hz notch filter. The signal was downsampled for analysis by a ratio of 4:1, thus cutting out all frequencies higher than 125 Hz. Only the last 1000 ms of each trial (250 samples) were examined in this study. This segment contained the interval during which the participant performed the imagination task. Based on a preliminary search for optimal length, it was established that the first half of each 1 second trial was enough for the analysis. No artifacts were removed or rejected, and no trial was discarded.

The trials of each data set were individually classified into the “stop” or “go” class using linear discriminant analysis (LDA). The classification accuracy was used as the criterion to compare the method with other methods in the literature. The classifications were carried out on single trials with 10-fold cross-validation to ensure the statistical significance of the results (Perreau Guimaraes, Wong, Uy, Grosenick, & Suppes, 2007). Prior to the classification, the number of spatiotemporal features was reduced substantially by selecting only the first 150 components obtained by principal
Figure 1: Contour maps for potential and scalp Laplacian distributions with three radial dipoles in the spherical head model. The brain potential is shown projected on the scalp surface. The Euclidean Laplacian was computed with $m = 3$.

component analysis (PCA). The cross-validation consisted of a single loop in which the 600 trials were partitioned into 10 distinct sets of 60 trials each. For each cross-validation step, one set was taken as the test set, and the union of the others formed the training set. Thus, 10 classification rates were generated per participant. The mean value of these rates determined the reported classification accuracy of the data.

3 Results

Figure 1 illustrates a simulation outcome using the Euclidean Laplacian. In this case, the estimation was in very good agreement with the analytic solution. The sources for this simulation were three current dipoles located at a distance of 6.2 cm from the center of the spheres. The ($\theta$, $\varphi$)-coordinates were at (75 degrees, 230 degrees), (65 degrees, 55 degrees), and (80 degrees, 100 degrees). The first dipole had polarization opposite to the others. The potential and Laplacian distributions were sampled using the locations of a geodesic net with 128 sensors (gray dots). The Euclidean mapping provided a much more reliable picture of the cortical activity than the scalp potential.

The goodness of the Euclidean Laplacian fitting was statistically evaluated by calculating the RMSE for single and multiple current dipoles. The estimates for single dipoles were made with the dipole at a fixed distance of $r = 7.8$ cm from the center of the spheres and with the polar and azimuth angles varying in the ranges of $\theta = 0$ to 90 degrees and $\varphi = 0$ to 360 degrees in steps of 1 degree. Hence, $91 \times 351 = 32,851$ estimates were performed in total. The distributions were sampled using arrays of $N = 64, 128, \text{and } 256$ electrodes, and the order of interpolation varied in the range $m = 3$ to 6. Table 1 presents the mean RMSE across all estimates. The errors were about the same order of magnitude for all conditions, $\sim 10^{-2} \mu V/cm^2$, although the estimates with $m = 3$, the lowest interpolation order, were slightly less accurate than the others in this case.

The fitting errors were mainly attributed to the difficulty in reconstructing the global peak of the analytic distribution. According to the discussion
Table 1: RMSE ($\times 10^{-2}\mu V/cm^2$) for the Fitting Provided by the Euclidean Approach for Single-Current Dipoles.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
<th>$m = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1.18</td>
<td>0.85</td>
<td>0.83</td>
<td>0.84</td>
</tr>
<tr>
<td>128</td>
<td>0.96</td>
<td>0.68</td>
<td>0.66</td>
<td>0.67</td>
</tr>
<tr>
<td>256</td>
<td>0.72</td>
<td>0.48</td>
<td>0.47</td>
<td>0.47</td>
</tr>
</tbody>
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Table 2: Rate of Success (%) in Locating the Current Dipole at its Nearest Electrode Site.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
<th>$m = 6$</th>
</tr>
</thead>
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<tr>
<td>64</td>
<td>97.54</td>
<td>95.97</td>
<td>95.55</td>
<td>94.50</td>
</tr>
<tr>
<td>128</td>
<td>98.07</td>
<td>97.91</td>
<td>97.18</td>
<td>96.79</td>
</tr>
<tr>
<td>256</td>
<td>97.57</td>
<td>97.82</td>
<td>98.06</td>
<td>97.86</td>
</tr>
</tbody>
</table>

In Appendix A, the global peak of the Laplacian occurs exactly at the dipole location, $(\theta', \varphi')$, where $\cos \gamma$, given by Equation A.7, is maximum. For a discrete distribution, this amplitude peak is expected to occur at the electrode site closest to $(\theta', \varphi')$. The success in achieving this result is a measure of reliability for the cortical mapping provided by the estimate. We calculated the percentage of the 32,851 Euclidean estimates for which the peak was at this exact location. The result is shown in Table 2. According to this criterion, the Euclidean mapping succeeded in correctly identifying the dipole location for at least 94.5% of the estimates.

The few errors in correctly identifying the dipole location occurred for dipoles that were at about the same distance to the two or three nearest electrode sites. In fact, we verified that the amplitude peak was at one of the two nearest electrode sites for more than 99.5% of the estimations, regardless of the $m$ and $N$ values. The map in Figure 2 illustrates this for the case of $m = 3$ and $N = 64$ electrodes. It is important to remark that the map in Figure 2 does not indicate scalp regions where the method is less accurate, but rather the locations where a current dipole is more likely not to be detected at its nearest electrode site.

This discussion has pointed out that the Euclidean Laplacian may not fit the analytic Laplacian perfectly, although it clearly succeeded in producing a reliable mapping of cortical activity. This fitting problem is more prominent at the peaks of the Laplacian distribution, where the radial current density is maximum. According to the histograms in Figure 3, fitting errors are more likely to occur for sources that produce relatively very small or

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5In this context, the distance between two points is defined as the great circle distance, which is the shortest distance between the two points along the spherical surface.
Figure 2: Scalp map showing the dipoles that were not localized correctly at its nearest electrode site for the case of \( m = 3 \) and \( N = 64 \) electrodes. Dots represent the dipoles and empty circles the electrode sites. The sites with negative \( z \)-coordinates are not shown in this figure.

Figure 3: Histogram for the amplitude of the peaks in the analytic and estimated Laplacian distributions.

very large peaks of amplitudes, and they significantly decrease by increasing the number of electrodes in the montage.

Figure 4 depicts a statistical analysis of fitting errors for Laplacian distributions generated by multiple cortical sources. The sources for this simulation were configurations of 1 to 21 radial dipoles (horizontal axis),
which were formed by linearly combining single dipoles. Since the surface Laplacian is a linear operation, the potential and Laplacian distributions for multiple dipoles were obtained by summing up the distributions for each dipole alone. We observe that the fitting errors, as measured by average RMSEs, significantly decreased with increasing electrode density. Hence, the more complex the source configuration is, the more advantageous it is to use a montage with a large number of sensors. For all conditions, the fitting error increased monotonically as the number of sources increased. The estimations with \( m = 5 \) and \( m = 6 \) were less accurate than those with \( m = 3 \) and 4, regardless of the number of sensors in the montage. The third-order interpolation provided the smallest RMSE for the montages with 64 and 128 electrodes, whereas fourth-order interpolation was the more accurate one for the montage with 256 electrodes.

The accuracy of the fitting in the Euclidean setting was compared to that of the spherical Laplacian for the same sources used in Figure 4. The result of this comparison is presented in Figure 5. The Euclidean Laplacian was computed with \( m = 3 \) and 4, whereas the spherical Laplacian was computed with the parameter \( m_s \) equal to 2 and 3. These \( m_s \)-values were selected because they optimize the spherical Laplacian estimate (see Babiloni et al., 1995). Similarly, the \( m \)-values were selected based on their performance.
Figure 5: RMSE of surface Laplacian estimations using the Euclidean and spherical approaches. The source configurations are the same as those in Figure 4.

Table 3: Mean Classification Rates (%) of the Experimental Data Across 10-Fold Cross-Validations.

<table>
<thead>
<tr>
<th>Method</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
<th>S7</th>
<th>S8</th>
<th>S9</th>
<th>S10</th>
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<tr>
<td>Potential</td>
<td>78.8</td>
<td>58.0</td>
<td>69.3</td>
<td>65.2</td>
<td>64.8</td>
<td>55.5</td>
<td>59.5</td>
<td>70.0</td>
<td>58.2</td>
<td>75.5</td>
<td>64.7</td>
</tr>
<tr>
<td>Discrete</td>
<td>81.7</td>
<td>56.2*</td>
<td>58.7</td>
<td>56.7**</td>
<td>56.0*</td>
<td>56.7**</td>
<td>64.2</td>
<td>74.0</td>
<td>55.5**</td>
<td>76.8</td>
<td>70.3</td>
</tr>
<tr>
<td>Spherical^a</td>
<td>83.0</td>
<td>60.0</td>
<td>68.7</td>
<td>68.7</td>
<td>64.3</td>
<td>59.2</td>
<td>67.5</td>
<td>78.7</td>
<td>59.8</td>
<td>78.8</td>
<td>70.7</td>
</tr>
<tr>
<td>Spherical^b</td>
<td>83.0</td>
<td>64.2</td>
<td>71.7</td>
<td>72.2</td>
<td>67.5</td>
<td>67.7</td>
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<td>88.8</td>
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<td>Euclidean^c</td>
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<td>Euclidean^d</td>
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<td>72.3</td>
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<td>73.3</td>
<td>78.3</td>
<td>66.5</td>
<td>81.0</td>
<td>71.8</td>
</tr>
</tbody>
</table>

*Significant at $p < 0.0015$; **significant at $p < 0.0006$; ***significant at $p < 0.00032$. Unstarred rates are significant at $p < 10^{-4}$.

^a$m = 2$; ^b$m = 3$; ^c$spherical scalp with $m = 3$; ^d$ellipsoidal scalp with $m = 4$.

shown in Figure 4. The result in Figure 5 shows that the predictions by both methods were in close agreement for all conditions. More precisely, the Euclidean estimate with $m = 3$ was close to the spherical estimate with $m_s = 2$, and the same held between $m = 4$ and $m_s = 3$. A small difference in favor of the spherical Laplacian method was observed for $N = 64$ electrodes, but this difference decreased quickly with increasing the electrode density.

Table 3 shows the rates obtained in the classification of the EEG signals. Each rate is a mean across the 10 rates generated in the 10-fold cross-validation. The classifications were performed using the raw signal
and four different methods of estimating the surface Laplacian. The discrete Laplacian consisted of approximating the Laplacian locally, using the formula (Hjorth, 1975)

$$\Delta_{\text{surf}}v_i \approx v_i - \frac{v_1 + v_2 + v_3 + v_4}{4},$$  \hspace{1cm} (3.1)

where $v_i$ is the potential at the electrode where the Laplacian is estimated and $v_1, v_2, v_3, v_4$ are the potentials at the nearest electrode sites. The spherical Laplacian was computed with truncating the summation over the Legendre polynomials $P_l(f \cdot f_i)$ at $\ell = 50$. The Euclidean methods were implemented using the minimum possible value for the $m$-parameter: $m = 3$ for the spherical scalp and $m = 4$ for the ellipsoidal scalp. The formula for the spherical scalp was discussed before and is given in detail in appendix B. Appendix C provides the formulas for the ellipsoidal model. Matlab code is available online as supplementary material.

We see that the mean rates all were above chance. No significant overall difference was observed between the 51 to 57 and 58 to 511 groups in terms of classification accuracy, despite their imagination tasks not being the same. The methods of the potential and the discrete Laplacian had similar performances. The most important exception occurred for participant 53, for whom a difference of more than 10% is observed in favor of the potential method. However, the discrete Laplacian performed better for 6 of the 11 participants. The rates of the spline-based methods were higher than the others, with the exceptions of participants 53, 55, and 511, for whom the spherical Laplacian was overcome by either the potential (53 and 55, $m = 2$) or the discrete Laplacian (51, $m = 3$). Similar to what occurred in the simulation, the spherical models, with geodesic and Euclidean norms, had almost the same performance.

An important remark about the result is that the spherical and the Euclidean Laplacians were implemented using regularization. That means that the rates in Table 3 are optimal values achieved by tuning the $\lambda$-parameter (Babiloni et al., 1995). The search for the best $\lambda$-value was carried out in the range 0.0001 to 0.3. The rates of the ellipsoidal method were improved further by optimizing the length of the ellipsoid's axes. Preliminary calculations showed that the ellipsoid model performs better if small values, rather than realistic values, are assigned to the axes. This reduces the diagonal elements of the matrices $K$ in equation 2.1, thus emphasizing the effect of the $\lambda$-regularization. In our computation, the axes were optimized by means of a random search in the range of lengths from 0.1 to 3.0 cm. Therefore, the remarkable performance of the ellipsoid method was achieved at the expense of an extra computation effort.

The significance of the result in Table 3 was estimated by computing the $p$-value under the null hypothesis of random labeling of the trials. For that
we used the binomial cumulative distribution function (Suppes, Perreau-Guimaraes, & Wong, 2009)

\[ P(R \geq \mu) = \sum_{k=\mu}^{T} \binom{T}{k} q^k (1-q)^{T-k}, \tag{3.2} \]

where \( T = 600 \) is the number of trials, \( \mu \) is the number of correctly classified trials, and \( q = 1/2 \) as the chance probability. The \( p \)-value corresponds to 1 minus the value given by this formula. For most of the rates, the null hypothesis was rejected at a significance level \( p < 10^{-4} \).

4 Conclusion

This work presented a framework for smooth EEG reconstruction and surface Laplacian estimations, based on splines in the Euclidean 3-space. This framework showed potential for wide application, including topographic mappings and brain-computer interfaces. The computation consists basically of linearly transforming the potential distribution. This facilitates analysis, speeds up computation, and makes it easy to integrate the method with other linear analysis methods, such as principal component analysis (Kayser & Tenke, 2006a, 2006b), and independent component analysis (Carvalhaes et al., 2009). The accuracy in estimating the surface Laplacian was demonstrated in the noiseless regime, using the widely used spherical scalp model. The simulation study showed that the method is capable of providing a reliable mapping of underlying cortical activity and that it performs quite close to the spherical Laplacian method for spherical geometries. Moreover, the Euclidean approach has the advantage of being given in a closed form, expressed in terms of a polynomial and translates of a well-behaved radial basis function. This avoids a fitting error due to truncations and greatly simplifies the computational implementation. The performance achieved by the ellipsoidal geometry in the study with empirical data demonstrated the usefulness of the method in EEG classification.

Appendix A: Radial Dipole in the Three-Sphere Model

The three-sphere model represents the human head as a system formed by three concentric spheres. The innermost sphere of radius \( r_1 \) and homogeneous conductivity \( \sigma_1 \) represents the brain. The intermediate and outer spheres of radii \( r_2 \) and \( r_3 \) and conductivity \( \sigma_2 \) and \( \sigma_3 \) model the skull and scalp. Consider the simple case of a static current \( i \) passing between two poles inside the brain, separated by a distance \( d \) along the z-axis (\( \theta = 0 \)). This current can be modeled by a static radial dipole located at a radial distance \( r' < r_1 \) from the center of the spheres and oriented in
the z direction. The electric potential $V^0$ generated by this dipole does not depend on the azimuthal angle $\varphi$. At any location $(r, \theta, \varphi)$, $r \in [r_{k-1}, r_k]$, where $k = 1, 2, 3$ stands for brain, skull, and scalp shells, respectively, and where $r_0 = 0$ is the center of the spheres, we have that (Nunez & Srinivasan, 2006, appendix G):

$$ V_k^0(r, \theta, \varphi) = \frac{i d}{4\pi \sigma_1 r^2} \sum_{n=1}^{\infty} \left[ A_{k,n} \left( \frac{r}{r_k} \right)^n + B_{k,n} \left( \frac{r_k}{r} \right)^{n+1} \right] n P_n(\cos \theta). $$

(A.1)

For instance, the potential $V_{\text{scalp}}^0$ at the scalp surface is the function defined by

$$ V_{\text{scalp}}^0(\theta, \varphi) = \frac{i d}{4\pi \sigma_1 r^2} \sum_{n=1}^{\infty} (A_{3,n} + B_{3,n}) n P_n(\cos \theta). $$

(A.2)

The coefficients $B_{1,n}$ are given by $B_{1,n} = (r'/r_1)^{n+1}$. The other coefficients are determined by the following boundary conditions:

$$ V_1^0(r_1, \theta, \varphi) = V_2^0(r_1, \theta, \varphi), $$

(A.3a)

$$ V_2^0(r_2, \theta, \varphi) = V_3^0(r_2, \theta, \varphi), $$

(A.3b)

$$ c_1 \frac{\partial V_1^0(r_1, \theta, \varphi)}{\partial r} = c_2 \frac{\partial V_2^0(r_1, \theta, \varphi)}{\partial r}, $$

(A.3c)

$$ c_2 \frac{\partial V_2^0(r_2, \theta, \varphi)}{\partial r} = c_3 \frac{\partial V_3^0(r_2, \theta, \varphi)}{\partial r}, $$

(A.3d)

$$ \frac{\partial V_3^0(r_3, \theta, \varphi)}{\partial r} = 0. $$

(A.3e)

That is, the potential and the radial current density are both continuous across the shells and the current density is 0 at the scalp surface. Let

$$ \sigma_{ij} = \frac{\sigma_i}{\sigma_j} \quad \text{and} \quad r_{ij} = \frac{r_i}{r_j}, \quad i, j = 1, 2, 3. $$

(A.4)

The problem of determining $A_{1,n}$, $A_{2,n}$, $A_{3,n}$, $B_{2,n}$, $B_{3,n}$ can be cast into the form

$$ \Lambda_n x_n = b_n, $$

(A.5a)
This calculation can be extended to the general case of a radial dipole randomly located inside the brain. Assume now that the dipole has coordinates \((r', \theta', \varphi')\) in spherical coordinates. In this case, the potential \(V_k(r, \theta, \varphi), r \in [r_{k-1}, r_k]\), at any location between two adjacent spheres \(k - 1\) and \(k\) is given by

\[
V_k(r, \theta, \varphi) = \frac{id}{4\pi \sigma_1 r'^2} \sum_{n=1}^{\infty} \left[ A_{k,n} \left( \frac{r}{r_k} \right)^n + B_{k,n} \left( \frac{r_k}{r} \right)^{n+1} \right] n P_n(\cos \gamma),
\]

where

\[
\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi')
\]  

(see Jackson, 1999, p. 110). A straightforward application of equation 2.10 yields the surface Laplacian on each shell. The expressions for the scalp potential and Laplacian are

\[
V_{\text{scalp}}(\theta, \varphi) = \frac{id}{4\pi \sigma_1 r'^2} \sum_{n=1}^{\infty} (A_{3,n} + B_{3,n}) n P_n(\cos \gamma),
\]

\[
\Delta_{\text{surf}} V_{\text{scalp}}(\theta, \varphi) = -\frac{id}{4\pi \sigma_1 r'^2} \sum_{n=1}^{\infty} (A_{3,n} + B_{3,n}) n^2 (n + 1) P_n(\cos \gamma),
\]

where the dependency on the azimuth angle \(\varphi\) is embedded in \(\cos \gamma\). The coefficients \(A_{i,n}\)’s and \(B_{i,n}\)’s depend only on the medium and remain given by equation A.5.

Appendix B: Formula for the Euclidean Laplacian on the Sphere

The Laplacian differentiation of the interpolant, equation 2.4, on a spherical surface can be performed using the identity (Jackson, 1999, p. 95)
\[ \Delta_{\text{surf}} v_\lambda(r, t) = \Delta v_\lambda(r, t) - \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} v_\lambda(r, t) \right]. \] (B.1)

After carrying out this differentiation, we replace the radial variable \( r \) by the scalp radius \( r_{\text{scalp}} \). First, we consider the term \( \Delta v_\lambda(r, t) \), which has the form

\[ \Delta v_\lambda(r, t) = \sum_{i=1}^{N} c_i \Delta \left( \| r - r_i \|^{2m-3} \right) + \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} d_{i+j+k+1} \Delta \left( x^{i-j} y^{j-k} z^k \right). \] (B.2)

Let \( \alpha \) be a generic variable representing any of the variables \( x, y, z \). Since

\[ \frac{\partial \| r - r_i \|}{\partial \alpha} = \frac{\alpha - \alpha_i}{\| r - r_i \|}, \] (B.3)

we have that

\[ \frac{\partial \| r - r_i \|^{2m-3}}{\partial \alpha} = (2m - 3) \| r - r_i \|^{2m-5} (\alpha - \alpha_i), \] (B.4a)

\[ \frac{\partial^2 \| r - r_i \|^{2m-3}}{\partial \alpha^2} = (2m - 3) \| r - r_i \|^{2m-5} \times \left[ (2m - 5) \| r - r_i \|^{-2} (\alpha - \alpha_i)^2 + 1 \right]. \] (B.4b)

Using these relations, we obtain the expression

\[ \Delta \left( \| r - r_i \|^{2m-3} \right) = (2m - 2)(2m - 3) \| r - r_i \|^{2m-5}. \] (B.5)

By differentiating the monomials \( x^{i-j} y^{j-k} z^k \), one obtains

\[ \Delta \left( x^{i-j} y^{j-k} z^k \right) = \left[ \frac{(i-j)(i-j-1)}{x^2} + \frac{(j-k)(j-k-1)}{y^2} + \frac{k(k-1)}{z^2} \right] \times x^{i-j} y^{j-k} z^k. \] (B.6)

Hence,

\[ \Delta v_\lambda(r, t) = (2m - 2)(2m - 3) \sum_{i=1}^{N} c_i \| r - r_i \|^{2m-5} + \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} d_{i+j+k+1} \times \left[ \frac{(i-j)(i-j-1)}{x^2} + \frac{(j-k)(j-k-1)}{y^2} + \frac{k(k-1)}{z^2} \right] \times x^{i-j} y^{j-k} z^k. \] (B.7)
Now we seek an expression for the radial derivatives in the right-hand side of equation B.1 in terms of the rectangular coordinates. Using the relations

\[
\frac{\partial}{\partial r} \frac{\|r - r_i\|}{\|r - r_j\|} = \cos \gamma_i, \quad (B.8a)
\]

\[
\frac{\partial}{\partial r} \cos \gamma_i = \frac{\hat{r} \cdot \hat{r} \frac{\|r - r_i\|}{\|r - r_j\|} - \hat{r} \cdot (r - r_j) \cos \gamma_i}{\|r - r_i\|^2} = \frac{\sin^2 \gamma_i}{\|r - r_i\|}, \quad (B.8b)
\]

where \( \hat{r} = r/\|r\| \) and \( \gamma_i \) is the angle between the vectors \( r \) and \( (r - r_i) \), we arrive at

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \|r - r_i\|^2 \right) = 2(2m - 3) \frac{\|r - r_i\|^2m-5}{\|r - r_i\|^2m-3}
\]

\[
\times \left[ \frac{\|r - r_i\|}{r} \cos \gamma_i + (m - 2) \cos^2 \gamma_i + \frac{1}{2} \sin^2 \gamma_i \right]. \quad (B.9)
\]

By expressing the monomials \( \phi_\ell(r) = x^{i-j} y^{j-k} z^k \) in the form

\[
\phi_\ell(r) = r^i (\sin \theta \cos \varphi)^{i-j} (\sin \theta \sin \varphi)^{j-k} \cos^k \theta,
\]

we readily obtain that

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \phi_\ell(r) \right) = \frac{i(i + 1)}{r^2} \phi_\ell(r). \quad (B.10)
\]

Finally, we obtain that the Laplacian on the spherical scalp surface is given by

\[
\Delta_{\text{surf}}u_\lambda(r, t) = 2(2m - 3) \sum_{i=1}^{N} \varepsilon_i \frac{\|r - r_i\|^3m-5}{r^{2m-3}}
\]

\[
\times \left[ m - 1 - \frac{\|r - r_i\|}{r_{\text{scalp}}} \cos \gamma_i - (m - 2) \cos^2 \gamma_i - \frac{1}{2} \sin^2 \gamma_i \right]
\]

\[
+ \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} \delta_{i+j+k+1}
\]

\[
\times \left\{ \left[ \frac{(i-j)(i-j-1)}{x^2} + \frac{(j-k)(j-k-1)}{y^2} + \frac{k(k-1)}{z^2} \right] \right\}
\]

\[
\times \phi_\ell(r) - \frac{i(i + 1)}{r^{2}_{\text{scalp}}} \frac{x^{i-j} y^{j-k} z^k}{x^2 + y^2 + z^2} \right\}. \quad (B.11)
\]
Note that this expression is discontinuous at the sampling points $r_i$'s for $m = 2$. For this reason, $m$ is required to be at least 3 for surface Laplacian estimations.

Appendix C: The Laplacian on Elliptical Scalp Models

Our goal here is to derive an expression for the surface Laplacian on an ellipsoidal surface in terms of the Cartesian coordinates $x, y, z$. We will assume, without loss of generality, that the scalp surface is described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

(C.1)

where the coordinates are labeled so that $a > b > c$. Our starting point is the ellipsoidal coordinate system, a three-dimensional orthogonal system that results from solving the following equation in terms of $x, y, z$:

$$\frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1.$$

(C.2)

The three nonequal real roots, $\rho, \mu, \nu$, of this cubic equation lie in the ranges

$$\rho \geq -c^2, \quad -c^2 \geq \mu \geq -b^2, \quad -b^2 \geq \nu \geq -a^2.$$

(C.3)

The surfaces of constant $\rho$ are ellipsoids, and the surfaces of constant $\mu$ and $\nu$ are hyperboloids of one and two sheets, respectively. In particular, the ellipsoid C.1 corresponds to $\rho = 0$. From equation C.2, the transformation from Cartesian coordinates to ellipsoidal coordinates is carried out by the system

$$x^2 = \frac{(\rho + a^2)(\mu + a^2)(\nu + a^2)}{(a^2 - b^2)(a^2 - c^2)},$$

(C.4a)

$$y^2 = \frac{(\rho + b^2)(\mu + b^2)(\nu + b^2)}{(b^2 - a^2)(b^2 - c^2)},$$

(C.4b)

$$z^2 = \frac{(\rho + c^2)(\mu + c^2)(\nu + c^2)}{(c^2 - a^2)(c^2 - b^2)}.$$

(C.4c)

The differentiation of the above expressions provides the scale factors $h_1, h_2, h_3$, which yield the element of length $ds$ in terms of the ellipsoidal coordinates—namely,

$$(ds)^2 = h_1^2(d\rho)^2 + h_2^2(d\mu)^2 + h_3^2(d\nu)^2,$$

(C.5)
where

\[ h_1 = \sqrt{\frac{(\rho - \mu)(\rho - \nu)}{4R(\rho)}}, \quad h_2 = \sqrt{\frac{(\mu - \rho)(\mu - \nu)}{4R(\mu)}}, \quad (C.6a) \]

\[ h_3 = \sqrt{\frac{(\nu - \rho)(\nu - \mu)}{4R(\nu)}}, \quad R(\xi) = (\xi + a^2)(\xi + b^2)(\xi + c^2). \quad (C.6b) \]

The general expression for the Laplacian is (Wang & Guo, 1989, p. 578)

\[ \Delta v_\lambda = \frac{1}{h_1h_2h_3} \left[ \frac{\partial}{\partial \rho} \left( \frac{h_2h_3}{h_1} \frac{\partial v_\lambda}{\partial \rho} \right) + \frac{\partial}{\partial \mu} \left( \frac{h_1h_3}{h_2} \frac{\partial v_\lambda}{\partial \mu} \right) + \frac{\partial}{\partial \nu} \left( \frac{h_1h_2}{h_3} \frac{\partial v_\lambda}{\partial \nu} \right) \right]. \quad (C.7) \]

Using equation C.6, we find that

\[ \Delta v_\lambda = \frac{4\sqrt{R(\rho)}}{(\rho - \mu)(\rho - \nu)} \frac{\partial}{\partial \rho} \left( \sqrt{R(\rho)} \frac{\partial v_\lambda}{\partial \rho} \right) \]
\[ + \frac{4\sqrt{R(\mu)}}{(\mu - \rho)(\mu - \nu)} \frac{\partial}{\partial \mu} \left( \sqrt{R(\mu)} \frac{\partial v_\lambda}{\partial \mu} \right) \]
\[ + \frac{4\sqrt{R(\nu)}}{(\nu - \rho)(\nu - \mu)} \frac{\partial}{\partial \nu} \left( \sqrt{R(\nu)} \frac{\partial v_\lambda}{\partial \nu} \right). \quad (C.8) \]

Hence, the Laplacian on the ellipsoidal surface \( \rho = 0 \) is simply

\[ \Delta_{surf} v_\lambda = \Delta v_\lambda \left[ \frac{4\sqrt{R(\rho)}}{(\rho - \mu)(\rho - \nu)} \frac{\partial}{\partial \rho} \left( \sqrt{R(\rho)} \frac{\partial v_\lambda}{\partial \rho} \right) \right]_{\rho=0} \]
\[ = \Delta v_\lambda \left[ \frac{2}{\mu \nu} \left[ \frac{dR(\rho)}{d\rho} \frac{\partial v_\lambda}{\partial \rho} + 2R(\rho) \frac{\partial^2 v_\lambda}{\partial \rho^2} \right] \right]_{\rho=0} \]
\[ = \Delta v_\lambda - 2a^2b^2c^2 \left[ \frac{1}{\mu \nu} \left[ \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \frac{\partial v_\lambda}{\partial \rho} + 2 \frac{\partial^2 v_\lambda}{\partial \rho^2} \right] \right]_{\rho=0} \quad (C.9) \]

We want to express this formula in terms of rectangular coordinates. The term \( \Delta v_\lambda \) was calculated before and is given by the expression in equation B.7. It follows from equation C.4 that

\[ \mu \nu \big|_{\rho=0} = \frac{b^2c^2}{a^2} x^2 + \frac{a^2c^2}{b^2} y^2 + \frac{a^2b^2}{c^2} z^2. \quad (C.10) \]
To handle the partial derivatives \( \partial v_\lambda / \partial \rho \) and \( \partial^2 v_\lambda / \partial \rho^2 \), we recall the formula
\[
\frac{\partial}{\partial \rho} = \frac{\partial x}{\partial \rho} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \rho} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \rho} \frac{\partial}{\partial z}.
\] (C.11)

It results from equations C.4 that the terms \( \partial x / \partial \rho \), \( \partial y / \partial \rho \), \( \partial z / \partial \rho \) in this formula are mapped onto the rectangular functions \( x/[2(\rho + a^2)] \), \( y/[2(\rho + b^2)] \), \( z/[2(\rho + c^2)] \), respectively. Hence,
\[
\frac{\partial v_\lambda}{\partial \rho} \bigg|_{\rho=0} = \frac{x}{2a^2} \frac{\partial v_\lambda}{\partial x} + \frac{y}{2b^2} \frac{\partial v_\lambda}{\partial y} + \frac{z}{2c^2} \frac{\partial v_\lambda}{\partial z}.
\] (C.12)

For the term \( \partial^2 v_\lambda / \partial \rho^2 \), we have that
\[
\frac{\partial^2 v_\lambda}{\partial \rho^2} \bigg|_{\rho=0} = \left\{ \left( \frac{\partial^2 x}{\partial \rho^2} \right) \left( \frac{\partial v_\lambda}{\partial x} \right) + \left( \frac{\partial x}{\partial \rho} \right) \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v_\lambda}{\partial x} \right) \right] + \left( \frac{\partial^2 y}{\partial \rho^2} \right) \left( \frac{\partial v_\lambda}{\partial y} \right) \right. \\
+ \left( \frac{\partial y}{\partial \rho} \right) \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v_\lambda}{\partial y} \right) \right] + \left( \frac{\partial^2 z}{\partial \rho^2} \right) \left( \frac{\partial v_\lambda}{\partial z} \right) \\
+ \left( \frac{\partial z}{\partial \rho} \right) \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v_\lambda}{\partial z} \right) \right] \right\} \bigg|_{\rho=0}
\] (C.13)
or
\[
\frac{\partial^2 v_\lambda}{\partial \rho^2} \bigg|_{\rho=0} = -\frac{x}{4a^4} \left( \frac{\partial^2 v_\lambda}{\partial x^2} \right) + \frac{x}{2a^2} \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v_\lambda}{\partial x} \right) \right]_{\rho=0} - \frac{y}{4b^2} \left( \frac{\partial v_\lambda}{\partial y} \right) \\
+ \frac{y}{2b^2} \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v_\lambda}{\partial y} \right) \right]_{\rho=0} - \frac{z}{4c^2} \left( \frac{\partial v_\lambda}{\partial z} \right) \\
+ \frac{z}{2c^2} \left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v_\lambda}{\partial z} \right) \right]_{\rho=0}.
\] (C.14)

According to equation C.4,
\[
\left[ \frac{\partial}{\partial \rho} \left( \frac{\partial v_\lambda}{\partial x} \right) \right]_{\rho=0} = \frac{x}{2a^2} \frac{\partial^2 v_\lambda}{\partial x^2} + \frac{y}{2b^2} \frac{\partial^2 v_\lambda}{\partial y^2} + \frac{z}{2c^2} \frac{\partial^2 v_\lambda}{\partial z^2},
\] (C.15)

with similar expressions for the partial derivatives \( \partial v_\lambda / \partial y \) and \( \partial v_\lambda / \partial z \). Carrying through the calculations, one finally obtains, after simplifications,
\[
\frac{\partial^2 v_\lambda}{\partial \rho^2} \bigg|_{\rho=0} = \frac{1}{4} \left\{ -\frac{x}{a^4} \frac{\partial v_\lambda}{\partial x} - \frac{y}{b^4} \frac{\partial v_\lambda}{\partial y} - \frac{z}{c^4} \frac{\partial v_\lambda}{\partial z} + \frac{x^2}{a^4} \frac{\partial^2 v_\lambda}{\partial x^2} + \frac{y^2}{b^4} \frac{\partial^2 v_\lambda}{\partial y^2} \\
+ \frac{z^2}{c^4} \frac{\partial^2 v_\lambda}{\partial z^2} + 2 \frac{xy}{a^2 b^2} \frac{\partial^2 v_\lambda}{\partial x \partial y} + 2 \frac{xz}{a^2 c^2} \frac{\partial^2 v_\lambda}{\partial x \partial z} + 2 \frac{yz}{b^2 c^2} \frac{\partial^2 v_\lambda}{\partial y \partial z} \right\}.
\] (C.16)
Thus, the formula for the Laplacian on an ellipsoidal surface is given by equations 9.7, 9.10, 9.12, and 9.16. The spatial derivatives of the interpolant $v_\lambda$ involved in these equations are computed using the following expressions:

\[
\frac{\partial v_\lambda}{\partial \alpha} = (2m - 3) \sum_{i=1}^{N} c_i(\alpha - \alpha_i) \|r - r_i\|^{2m-5}
\]

\[
+ \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} d_{i+j+k+1} \frac{\partial}{\partial \alpha} \left(x^{i-j-1} y^{j-k} z^k\right),
\]  

(C.17a)

\[
\frac{\partial^2 v_\lambda}{\partial \alpha^2} = (2m - 3) \sum_{i=1}^{N} c_i(\alpha - \alpha_i) \|r - r_i\|^{2m-7} \left[(2m-5)(\alpha - \alpha_i)^2 + \|r - r_i\|^2\right]
\]

\[
+ \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} d_{i+j+k+1} \frac{\partial^2}{\partial \alpha^2} \left(x^{i-j-2} y^{j-k} z^k\right),
\]  

(C.17b)

\[
\frac{\partial^2 v_\lambda}{\partial \alpha \partial \beta} = (2m - 3)(2m - 5) \sum_{i=1}^{N} c_i(\alpha - \alpha_i)(\beta - \beta_i) \|r - r_i\|^{2m-7}
\]

\[
+ \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} d_{i+j+k+1} \frac{\partial^2}{\partial \alpha \partial \beta} \left(x^{i-j-2} y^{j-k} z^k\right),
\]  

(C.17c)

where $\alpha$ and $\beta$ can be any of the variables $x, y, z$ with $\alpha \neq \beta$. Note that the continuity of the Laplacian at the sampling points $r_i$'s requires that $2m - 7 \geq 0$, that is, the parameter $m$ must be greater than or equal to 4 for Laplacian estimations on ellipsoidal scalp models.

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