ACCELERATED PROGRAM IN ELEMENTARY-SCHOOL MATHEMATICS--THE SECOND YEAR

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Project Director
Patrick Suppes

Staff: Miss Diana Axelsen
Miss Phyllis Cole
Mrs. Eleanor Cooper
Mrs. Peggy Coulter
Mrs. Muriel LaAnyne
Mr. Joseph Marchal

Institute for Mathematical Studies in the Social Sciences
Stanford University
Stanford, California

ACCELERATED PROGRAM IN ELEMENTARY-SCHOOL MATHEMATICS--THE SECOND YEAR*

1. Introduction

This report describes the second year of the accelerated program in elementary-school mathematics being conducted by the Institute for Mathematical Studies in the Social Sciences. A description of the first year of the study, including details of the procedures by which the students were selected, will be found in "Accelerated program in elementary-school mathematics--the first year," Psychology in the Schools, Vol. 2, (1965), pages 195-203. The present report is written to be as homogeneous as possible with the description of the first year of the study. The second section of this report describes the curriculum content of the second year, including both the work done during the academic year 1964-65 and the summer session of July 1965. The third and fourth sections give a brief description of the class composition and class procedures. The fifth section on results describes the systematic behavioral data we have collected and analyzed.

The tables and figures are similar to those presented in describing the first year's work. As has been the experience of most people working in longitudinal curriculum studies, it is impossible to describe all the results in systematic form. For example, in the work with finite symmetry groups in the summer of 1965, systematic behavioral data were not collected, and consequently anecdotal descriptions of student reactions and progress are contained in the section on curriculum content.

During the summer session the students were introduced to the Computer-based Instruction Laboratory at Stanford and were given work in logic and arithmetic drills. Description of this new phase is contained both in the sections on curriculum content and on results.

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Finally, it should be remarked that the group for 1964-65 consisted of 34 children who were bright second graders in the second year of an accelerated program in elementary-school mathematics. This fact should be borne in mind in reading the description of curriculum content.

2. Curriculum Content

A. Academic Year, 1964-65

To develop a basic background in mathematics the children continued to work individually in the Sets and Numbers text material. The same concepts introduced in the material last year were now extended to present more difficult concepts such as the use of the laws of arithmetic to teach carrying in multiplication.

This year more stress was laid on enriching the basic material. Some of this supplementary material was especially written for these children. Each child worked through a programmed instruction booklet written on the history of numbers by a member of the project. Babylonian, Egyptian, and Roman systems of numeration were introduced. The material showed the numerals for each system and gave the child simple problems to work out. The number of pages and problems completed varied with each child.

The topic of coordinate systems was introduced with the game of tic-tac-toe played in the first quadrant. The terms 'origin', 'horizontal axis', 'vertical axis', 'coordinate', and 'ordered pair' were introduced. The children quickly became familiar with the notion of an ordered pair, and the game was extended to all four quadrants. The children were told to read "-5" as "negative five". They quickly found that in quadrant 3, both coordinates were negative, that a point lying on the vertical axis has 0 as the horizontal coordinate, etc. The mastery of the strategy of the game was quite rapid by all the groups. After the game, the children went on to plotting points, labeling them, and drawing given line segments to form figures such as a box, a star, and a pentagon. Given a linear equation in two variables, the children were able to make a table of values, so long as positive values were involved. These points were then plotted. Most of the children were easily able to extrapolate to quadrants 2 and 3 when asked, "Where must the next point be if all the points are to lie on the same line?"

The groups did two special geometry sections--axes of symmetry and planes of symmetry. This material was written for the sixth-grade Sets and Numbers book, but was an excellent follow-up to the work with lines of symmetry which most of the children had had in the 1964 summer school.

Using familiar geometric figures as symbols, the children quickly caught the idea of counting in base three. Later the children filled in a dittoed sheet involving counting in base four (again, geometric figures were used as symbols). To find simple sums in the number bases an addition table was used.

Two games were used to introduce addition of signed integers. The games were taken from materials of the Madison Project, Discovery in Mathematics, by Robert B. Davis (published by Addison Wesley). The "Postman" game met with more success than the game "Pebbles in the Bag," although the latter game involved the children moving about, counting objects, and in general participating more actively than in the "Postman" game.

The logic of identity was used to introduce the concept of proof. The vocabulary introduced included 'theorem', and 'axiom'. Many of the children were already familiar with the concepts of commutativity and associativity from Sets and Numbers, Book 3B. Some of the theorems proved included $3 = 1 + 2$, $4 = 1 + (1 + 2)$, and $4 = 2 + 2$. For example, the proof that $4 = 2 + 2$ depended on the associative law for addition, and the definitions $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$.

Proof:	$4 = 3 + 1$	Definition of 4
	$= (2 + 1) + 1$	Definition of 3
	$= 2 + (1 + 1)$	Associative Law
	$= 2 + 2$	Definition of 2 .

Beginning in January, the children were given daily drills to provide appropriate maintenance of skills. The drills took several forms--some as dittoed material, 10 to 18 problems read by the teacher, and games. The games included "Twenty Questions," "Find the Rule," and carrying in mental addition.

B. Summer Session, July, 1965

Twenty-six children attended a four-week summer session at the Computer-based Laboratory on the Stanford University campus. The children were divided into three groups, with each group coming for an hour. Each child spent half the period working in the classroom on geometry; the other half was spent at computer-based terminals working on logic and arithmetic drills.

Classroom Program: Finite Symmetry Groups. During the summer program the children studied finite symmetry groups. They worked in small groups using specially prepared material. The children began with a review and extension of last summer's work on lines of symmetry. They then investigated geometric figures which can be put in several positions without changing their appearance. Many objects can be put into different positions without substantially altering their appearance--turning a window pane upside-down in its frame does not change the appearance of the window, a fan looks the same in several positions. Any such object can serve as a model on which to base a symmetry group. The symmetry groups are a particularly appropriate introduction to group theory because of the large number of such readily-available models. The figures used in the material include triangles, squares, pentagons, pictures of pinwheels, propellors, etc. The children found that there is a single efficient motion which will move a figure to any given position; they learned how many of these motions (functions) there are for a given figure, and how to symbolize the motions.

The close relationship between the group for a figure and the lines of symmetry for the figure is stressed throughout the material. The children soon learned, however, that a group is more than just a set of motions. They found that there is a natural way of "adding" the motions in a group. (The addition of two motions was explained quite simply as the process of doing first one motion and then the other.) After some practice adding motions, the children were shown that they could equate the sum of two motions with a single more efficient motion that produces the same result. The addition facts for groups were summarized in group addition tables (Cayley Tables). Since all of the properties of the finite symmetry groups are present in the smaller groups-- none of the groups used in the work have more than 12 elements and most have less than 8--the addition tables are small, and tedious computations can be avoided. Commutativity was reviewed and applied to group addition. The children already had a good idea of what commutativity is. The material contains examples of both commutative and non-commutative additions, and the children learned how to distinguish commutative from non-commutative groups by inspection of the group addition tables. Throughout the material the existence of a group identity (a "zero") is stressed and the addition properties of the identity are investigated. Finally the children were shown that there are some motions which are so versatile that they can do the work of all the other motions in the group; the existence of such motions (generators) is closely connected with commutativity in the symmetry groups.

The study of symmetry groups can serve the children in two ways. First, it can reinforce many concepts to which they have been previously introduced. Second, it can provide them with a background for ideas yet to come. Besides extending their knowledge of lines of symmetry, the children saw many of the other symmetry properties of the regular polygons--equality of sides and angles, positions of centroids, etc. Up to this time the only sets the children had met were sets of physical objects or, occasionally sets of numbers; sets of functions (motions) provide a valuable example for additional work with sets.

Work at the computer-based terminals. The work on the terminals was of two types: first, logic; and second, computation, using simple number facts. The daily computation drill usually consisted of 30 problems. The problem appeared on a cathode-ray tube, as $4 \times 5 =$ ____; the child then had 20 seconds to type in the correct answer before "time is up" appeared. If no response was made in 20 seconds, the child was then given the same problem. If there was again no response within 20 seconds, "time is up" appeared and the child was given the correct answer to type in. The problem was then repeated. If there was no response on this third trial, he was given the time-up signal and the answer, and then sent to the next problem. Essentially the same procedure was followed when the child made an error. Each child did at least one drill every day.

For logic work the 26 children were divided into two groups--one group working with English sentences only, and the other using symbolization after Lesson 5. Lessons 1 and 2 used a simple story to give an intuitive understanding of simple single-step applications of modus ponens. The visual problem format appearing on the cathode-ray tube (CRT) was explained by the audio that went along with it. Each problem was answered by typing the correct letter from multi-choice answers. Here is an example of what appeared on the CRT.

1. If the gate is open, then Spot ran away.
2. If the little door is open, then Spot is under the house.
3. The gate is open.

A. Spot is under the house.
B. Spot ran away.
C. Spot is sleeping under the tree.

The child had only to make the correct choice of A, B, or C. Lessons 2 - 6 followed the same format with a gradual build-up of vocabulary, and more difficult problems. Lesson 3, for example, introduced the use of premise numbers, and Lesson 5 introduced two-step problems. Lesson 6 introduced the first explicit rule of inference, modus ponens. For the students this was called the IF rule. At this point the student had to type in the name of the rule used instead of A, B, or C. Each lesson had a branch for the children who did not meet the criterion on the main line. While most children at least met the criterion on the branch, a teacher-call was set up for those who did not. The child then went through the concepts of the lesson with the teacher before trying the branch lesson again at the terminal.

Each of the six lessons in logic is fairly long. Although additional lessons had been prepared and programmed for the computer, there was not time during the summer session to use them, but the work in logic is continuing during 1965-66.

3. Group Composition

With the start of classes in September, 1964, there were 32 children in the group--15 girls and 17 boys. The distribution among the four schools these children attended was 6, 7, 8, and 11.

At the request of the school district, an exception was made and two more children were taken into the group. One was the child who had withdrawn in March, 1964. The other was a transfer into the district. The transfer student had shown exceptional potential when tested by the school psychometrist. His I.Q. placed him above the top of the I.Q. range of the group. A member of the project staff did the necessary testing to verify the high mathematical potential of the child.

4. Class Procedures

The teaching staff continued individual work with the Sets and Numbers texts, but group work was done for most of the supplementary material. Daily logs recording the problems attempted and errors committed were maintained through Sets and Numbers, Book 3B. Starting with Sets and Numbers, Book 4A, tests were given as each child finished a section of the book.

During the academic year 1964-65, classes were held five days a week with the time in some classes being increased to forty minutes. Classes continued to meet separately with an individual instructor in each of the four schools from which the students are drawn, except that during the summer session the students were brought to the Stanford University campus in three groups with approximately nine students in each group.

5. Results

The 1964-65 results again indicate a consistently high daily rate of performance. Figures 1 - 4 depict the overall group performance for the 36 weeks of work. On each figure, the number of students on which each biweekly group mean is based is shown along the top. Figures 1 and 2 give a biweekly cumulative record of problems and responses

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Insert Figures 1 and 2 about here

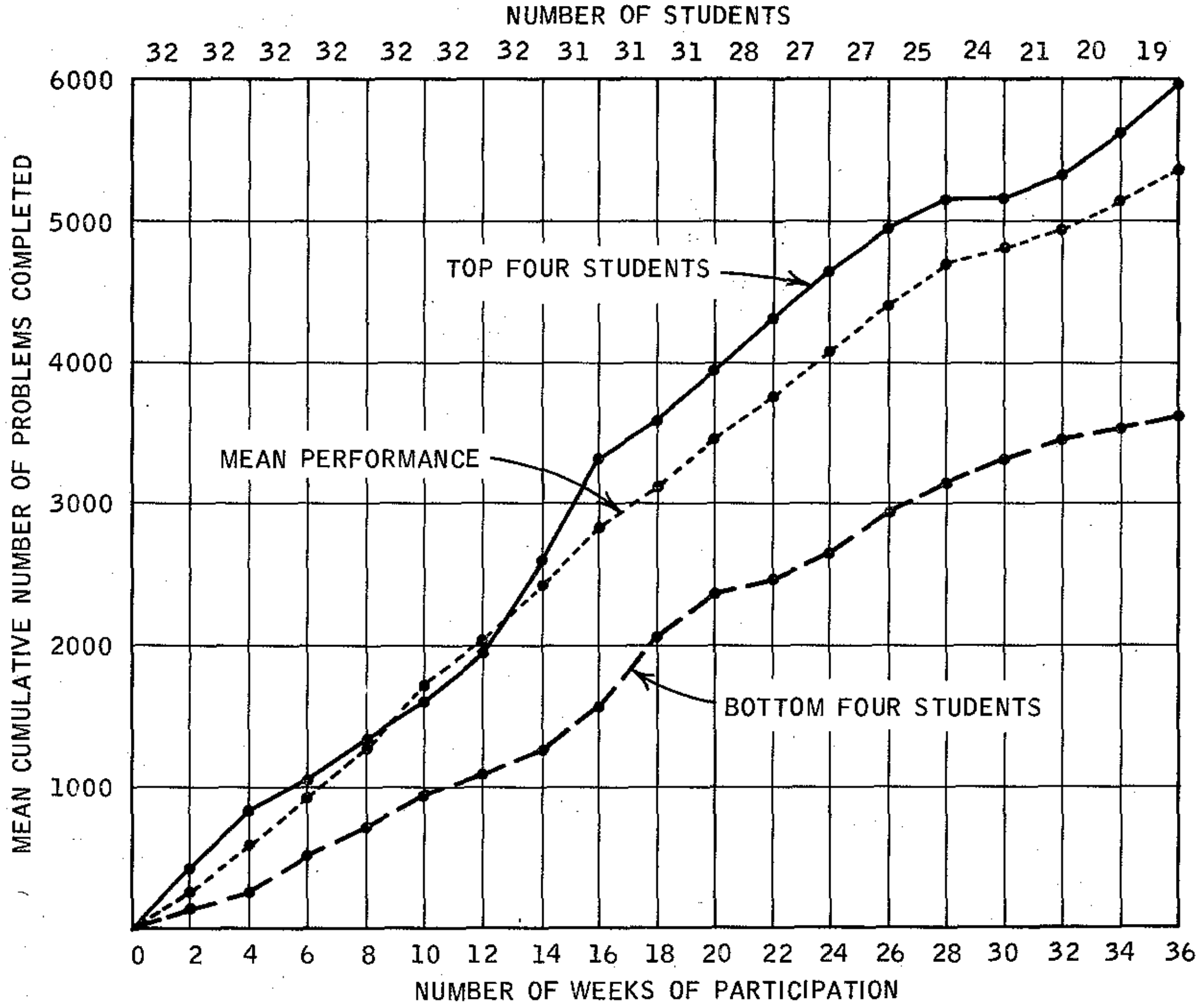

Figure 1. Curriculum acquisition curves in terms of number of problems.

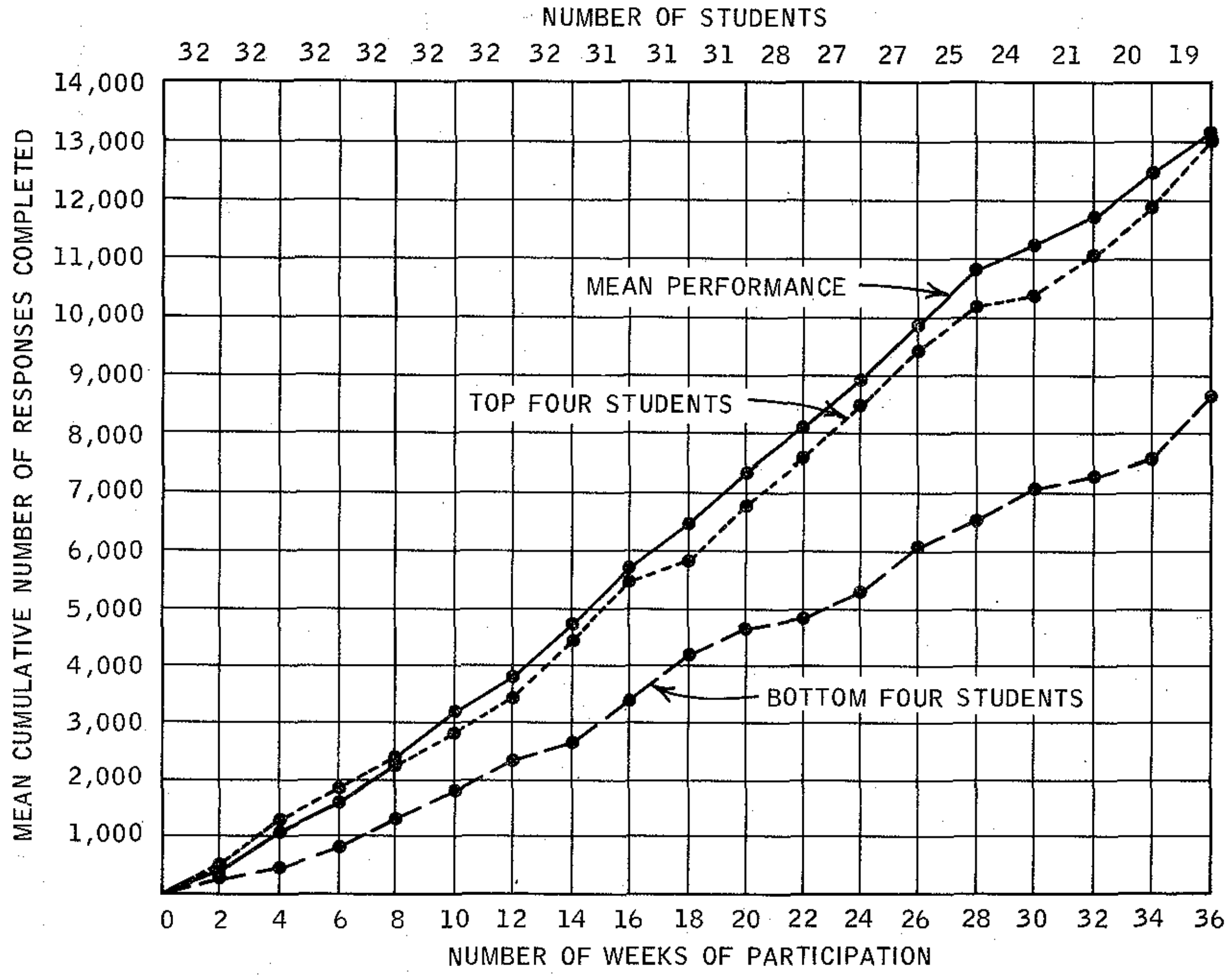

Figure 2. Curriculum acquisition curves in terms
of number of responses.

completed. On Figure 1, a calibration of the number of problems in terms of the Sets and Numbers text material is shown on the ordinate. The mean number of problems completed for 36 weeks of work was 4869.0. (This figure is based on 36-week totals for the 32 students who were in the group in September, 1964.) The ordinate of Figure 2 shows a calibration of the number of responses. The number of responses required for the completion of a single problem varied throughout the books. For example, in Books 2A and 3A, one problem often required only one or two responses, whereas many problems in Book 3B had several steps and required eight or nine responses. The mean acquisition rate was approximately one and one-eighth years of the curriculum. It is to be emphasized that this acquisition rate is for curriculum material written for students at least one grade level higher than the students in this study.

Table 1 presents a non-cumulative biweekly record of problem acquisition and error rates for the group. Figure 3 depicts the mean per-

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Insert Table 1, Figure 3 and 4 about here

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centage of problems in error for each two-week work period. The mean biweekly error percentage for the year was 4.5, an increase of 1.7 over last year. (The figure of 4.5 is based on biweekly mean error percentages for all 32 students on the basis of their work in Sets and Numbers, Book 3B.) Figure 4 shows the mean error percentages for responses.

Group acquisition rates can also be computed according to sections in the text materials. In Table 2, these results are presented for Books

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Insert Table 2 about here

3A and 3B. Many of the students began work in Book 2. Thus, not all students completed Book 3, and one did not reach the second section of 3A. As the last column in Table 2 shows, the results for 3B reflect decreasing numbers of students, since only the more proficient students worked in this book.

Only half of the students began work in Book 4A, and only half of these progressed beyond Section 3. No biweekly record of problems and

Table 1. Group Acquisition and Error Rates for Two-Week Periods

Weeks	Mean Number of Problems Completed	Standard Deviation	Mean Number of Problems in Error	Standard Deviation	N
1-2	236.3	137.8	7.0	6.5	32
3-4	334.3	304.3	6.5	5.8	32
5-6	375.0	246.4	9.8	7.3	32
7-8	317.2	208.1	8.8	6.6	32
9-10	476.7	304.2	11.5	8.7	32
11-12	283.0	260.2	9.7	8.5	32
13-14	402.8	282.8	12.7	7.2	32
15-16	380.4	248.0	12.5	8.5	31
17-18	338.1	193.6	11.1	9.2	31
19-20	317.2	226.8	10.0	9.1	31
21-22	302.5	195.0	10.1	7.4	28
23-24	323.9	216.3	13.2	8.8	27
25-26	314.9	153.3	14.7	8.3	27
27-28	301.4	270.1	10.6	6.9	25
29-30	109.2	139.0	6.7	9.5	24
31-32	167.2	70.7	10.3	7.6	21
33-34	205.6	139.6	11.3	11.1	20
35-36	180.8	143.2	11.8	8.5	19

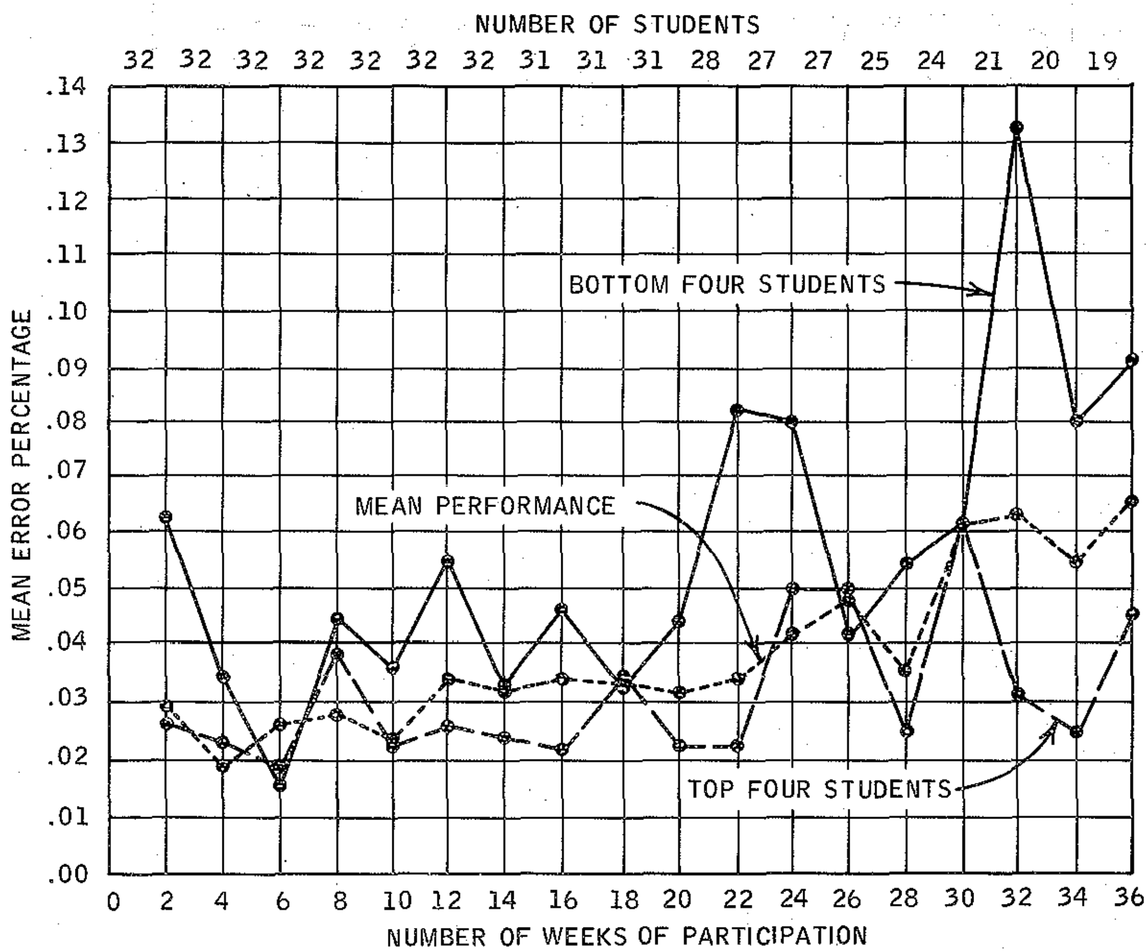

Figure 3. Mean percentage of problems completed which were in error.

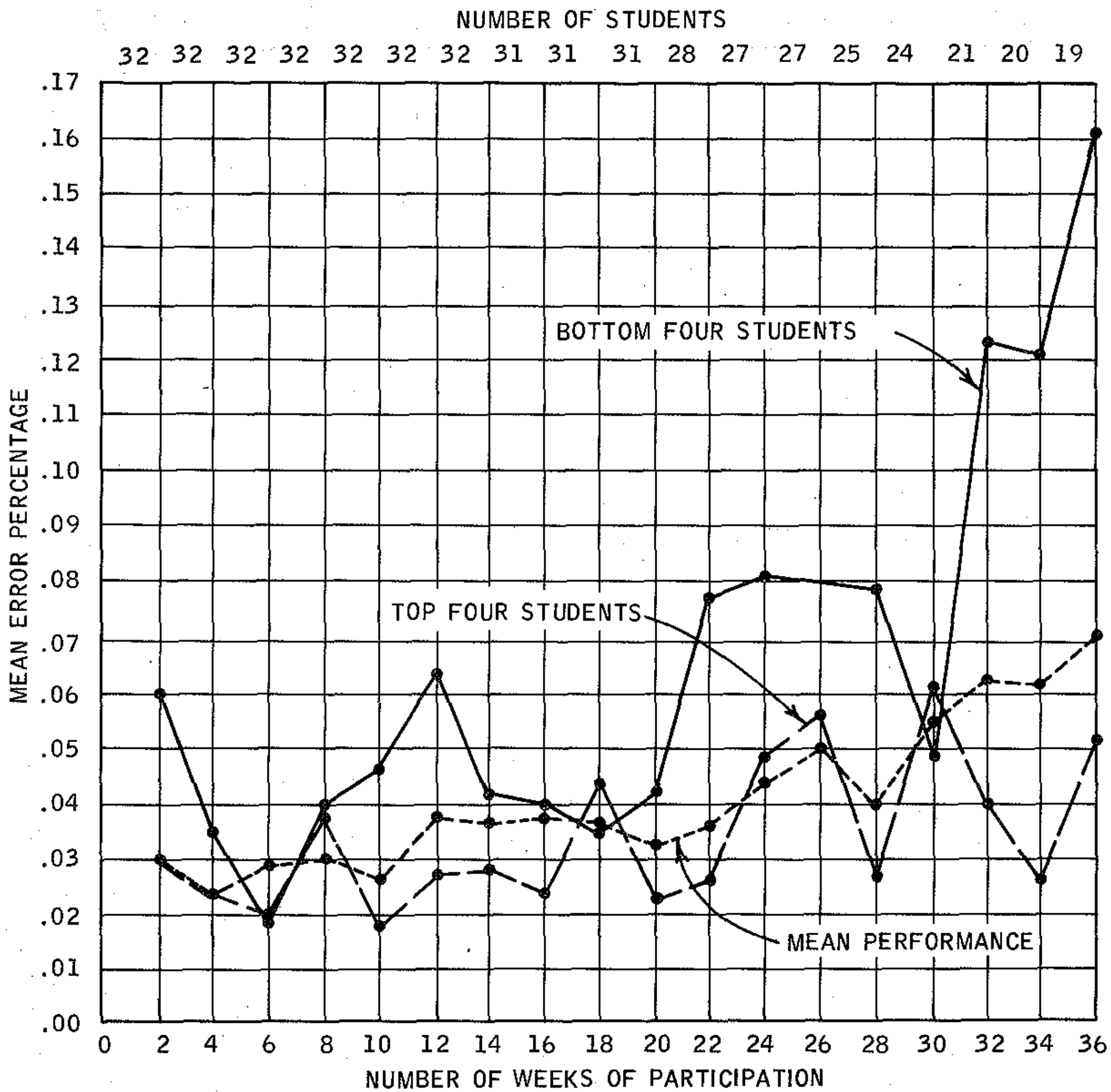

Figure 4. Mean percentage of responses completed which were in error.

Table 2. Daily Acquisition and Error Rates for Various Sections in the Sets and Numbers Text Material

Book Section	Mean No. of Problems Performed Daily	Standard Deviation	Mean No. of Problems in Error	Standard Deviation	N
Book 3A					
1. Sets	151.7	65.2	2.5	1.5	32
2. Multiplication	131.0	75.7	2.6	1.6	31
3. Carrying and Borrowing	73.5	48.6	2.7	1.6	31
4. Review	83.1	64.5	3.0	3.5	31
5. Division and More Multiplication	83.4	60.0	2.7	3.4	31
6. Fractions	69.3	33.1	1.9	1.1	31
7. Subsets and Less Than	63.8	27.7	1.8	1.0	31
8. Geometry	64.0	27.3	1.7	1.1	31
Book 3B					
1. Thousands, Hundreds, Tens, and Ones	46.0	30.4	2.0	.96	30
2. Geometry	44.9	39.0	1.6	1.1	26
3. More about Sets	61.7	35.8	1.5	1.1	26
4. Commutative Laws	80.5	33.5	1.5	1.1	26
5. Associative Laws	64.5	33.8	2.1	1.5	25
6. Shapes and Sizes	63.0	40.0	2.0	2.0	25
7. Distributive Law for Multiplication	41.2	16.3	1.8	.9	25
8. Lines of Symmetry	56.6	27.1	2.1	2.2	21
9. Word Problems	48.0	41.7	2.4	1.6	21

responses completed was kept for work done in 4A. However, Table 3 shows

Insert Table 3 about here

the mean percentage of problems in error for these sections. The groups on which the mean error percentages were based include students who did not complete the last section in which they were working.

The results this year again show great variation in acquisition and error rates for individual students. In Figure 1, mean cumulative curves for the top four and bottom four students are shown in addition to the group curve. Only students with a biweekly record for the entire 36 weeks of work were considered in selecting the top four and bottom four students. Thus, students working in Book 4A were not included. At the end of the year, these two groups of students were separated by slightly over one-half year of the curriculum. This difference is considerably less than the one and three-fourths years of the curriculum which separated the single fastest student from the single slowest student last year, but the reduction in spread is not surprising because groups rather than individuals were considered, and the many students working in Book 4A were not considered. The bottom four in the comparison group were not the four students who completed the fewest number of problems. Their ranks in the group according to number of problems completed were 25, 28, 29, and 31. The top four students, however, were ranked 1, 2, 3, and 4 in the entire group. (Their high ranks in comparison with the students working at a more advanced level in Book 4A is undoubtedly due to the greater difficulty of the more advanced material.) The mean number of problems completed for the year by these top four students was 6106.8; by the bottom four, 3606.3. Figure 3 shows the mean percentage of problems in error for these two groups. The bottom four students had a higher error percentage than the top four, for all but four of the two-week periods. The slower students had a mean error percentage of 6.3 for the year, compared to 3.6 for the faster students. The same tendencies can be seen in Figures 2 and 4, which show response acquisition and error rates for the same two groups of four students.

Table 3. Error Rates by Section for Sets and Numbers, Book 4A

Book Section	Mean Percentage of Problems in Error	Standard Deviation	N
Book 4A			
1. Review of Sets, Addition, and Subtraction	3.4	1.8	16
2. Review of Geometry	6.4	4.1	16
3. Addition and Subtraction	5.0	2.3	16
4. Geometry	11.0	4.6	8
5. Review of Multiplication and Division	4.4	3.2	7
6. Laws of Arithmetic	2.4	1.5	6
7. Shapes and Sizes	5.6	5.4	6
8. Distributive Law for Multiplication	3.7	3.1	5
9. Applications	9.4	---	2
10. Distributive Law for Division	2.5	---	1

Individual differences can also be seen by comparing the performances of the top half and bottom half of the students, but, as we would expect, the spread here is not so striking. These two groups were separated by slightly more than one-fourth year of the curriculum at the end of 36 weeks. These figures are based only on work done through Book 3B. However, there were 16 students who worked beyond this book. These children were initially included in the top and bottom groups, though they were dropped out in pairs as they finished 3B, (one from the top half and one from the bottom half). In Figures 5 - 8, the number of students in

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Insert Figures 5 - 8 about here

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each half on which the mean was based is shown along the top, for each two-week period. Figures 5 and 6 give biweekly cumulative results for mean number of problems and responses completed. The mean error percentages are shown in Figures 7 and 8.

The standard deviations given in Tables 1 and 2 also reflect strikingly large individual differences; the standard deviations represent over 60 per cent of the value of the mean except for a few sections in the text material. Similar variation was seen in the results for last year. However, Table 2 does show a lower standard deviation for geometry in Book 3A, compared to the deviations for Book 3B and for all the work in geometry last year. This section of Book 3A deals with the concepts of triangle, quadrilateral, pentagon, circle, concave and convex figures, and being inside, outside, and on a figure. The error rate was lower than that for the geometry work last year, although it was slightly greater than the rate for the geometry work in Book 3B.

When the entire group is considered, the curriculum separation between the top and bottom student is approximately two years. The fact that the difference obtaining at the end of the first year has not continued to increase substantially is significant. The probable explanation is the definitely increased difficulty and "density" of problems at the fourth-grade level where the more advanced students were working. The great emphasis on the algorithms of multiplication and long division in the upper elementary grades is not in all likelihood a desirable emphasis

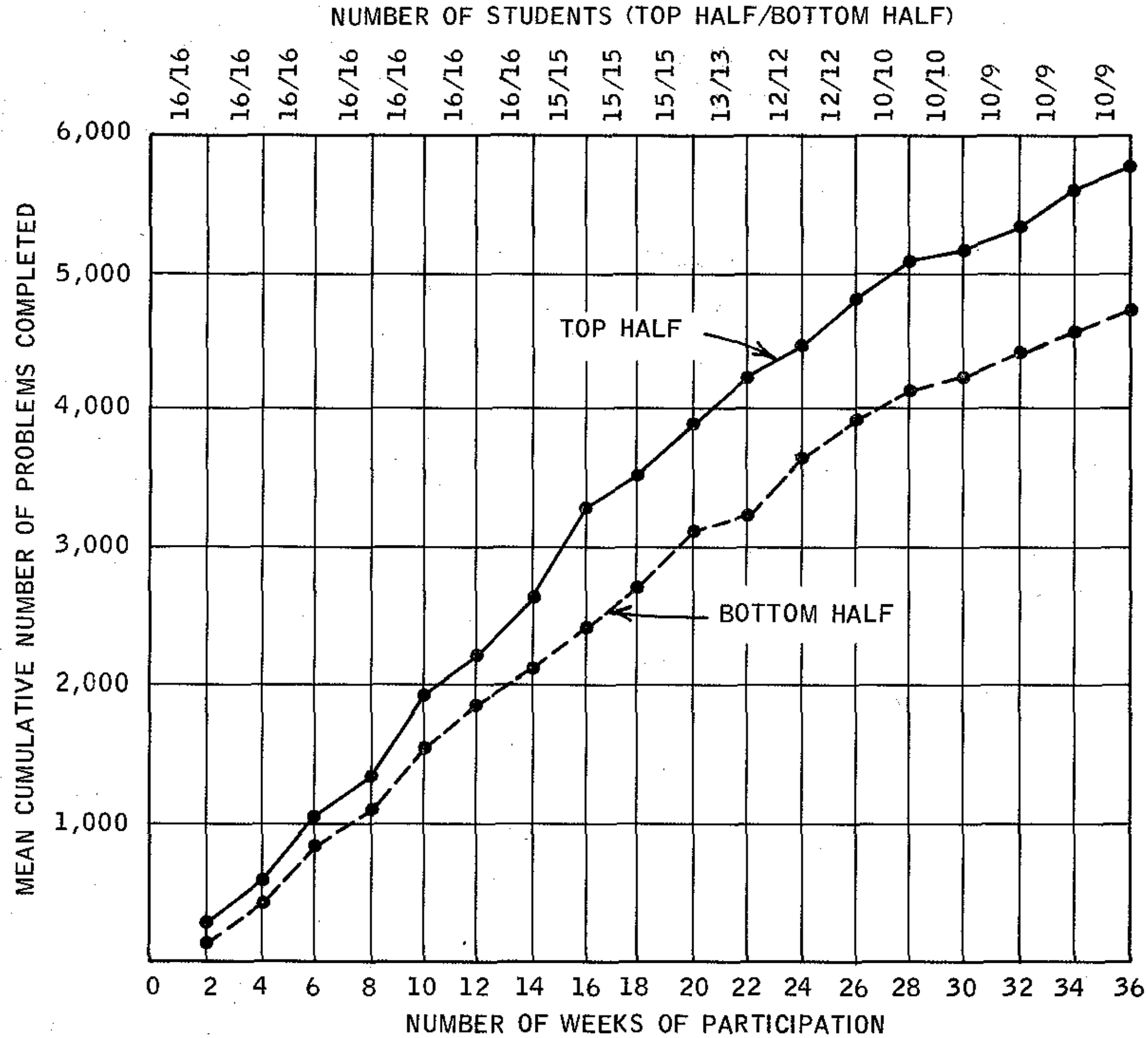

Figure 5. Curriculum acquisition curves in terms of number of problems.

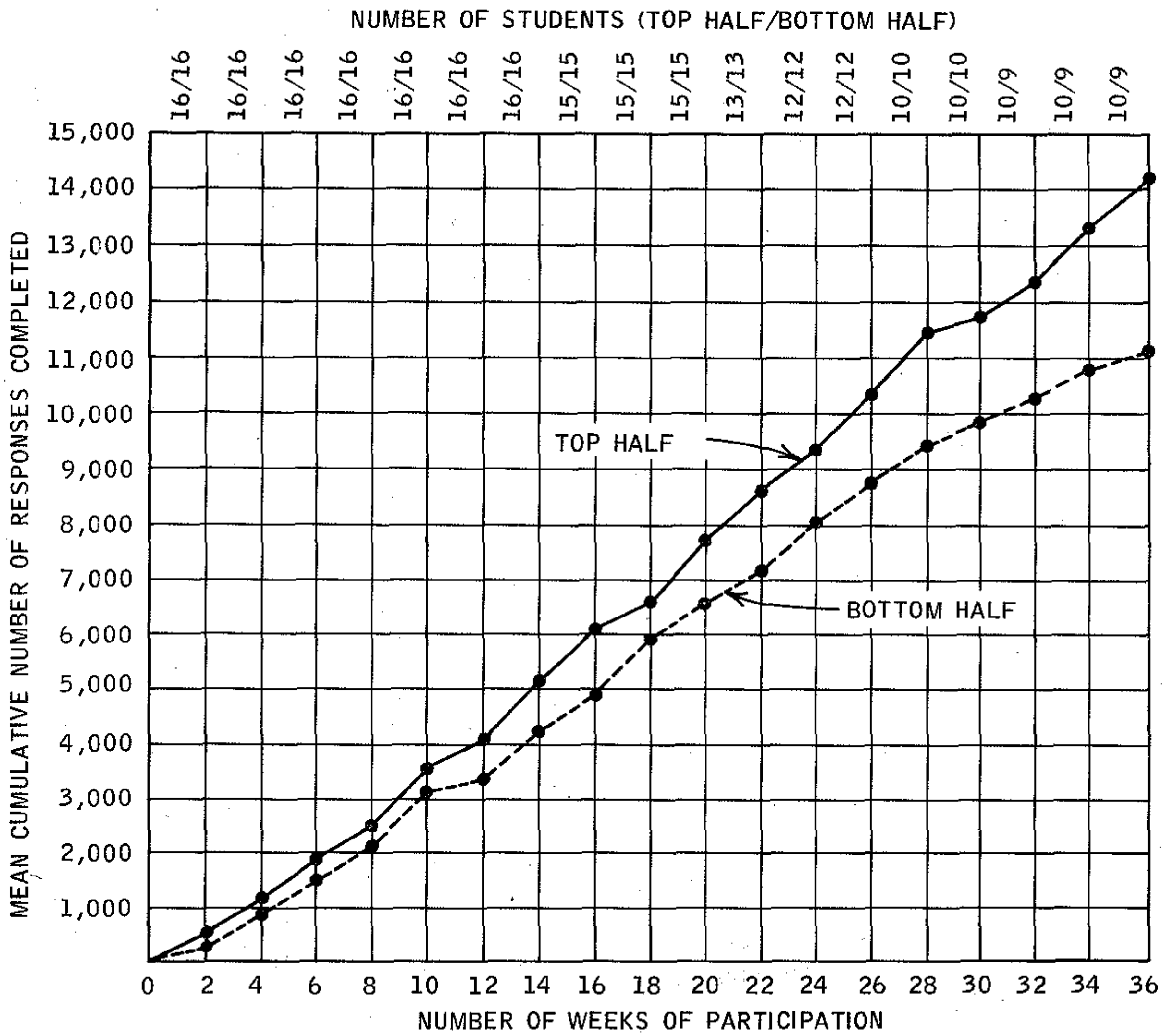

Figure 6. Curriculum acquisition curves in terms of number of responses.

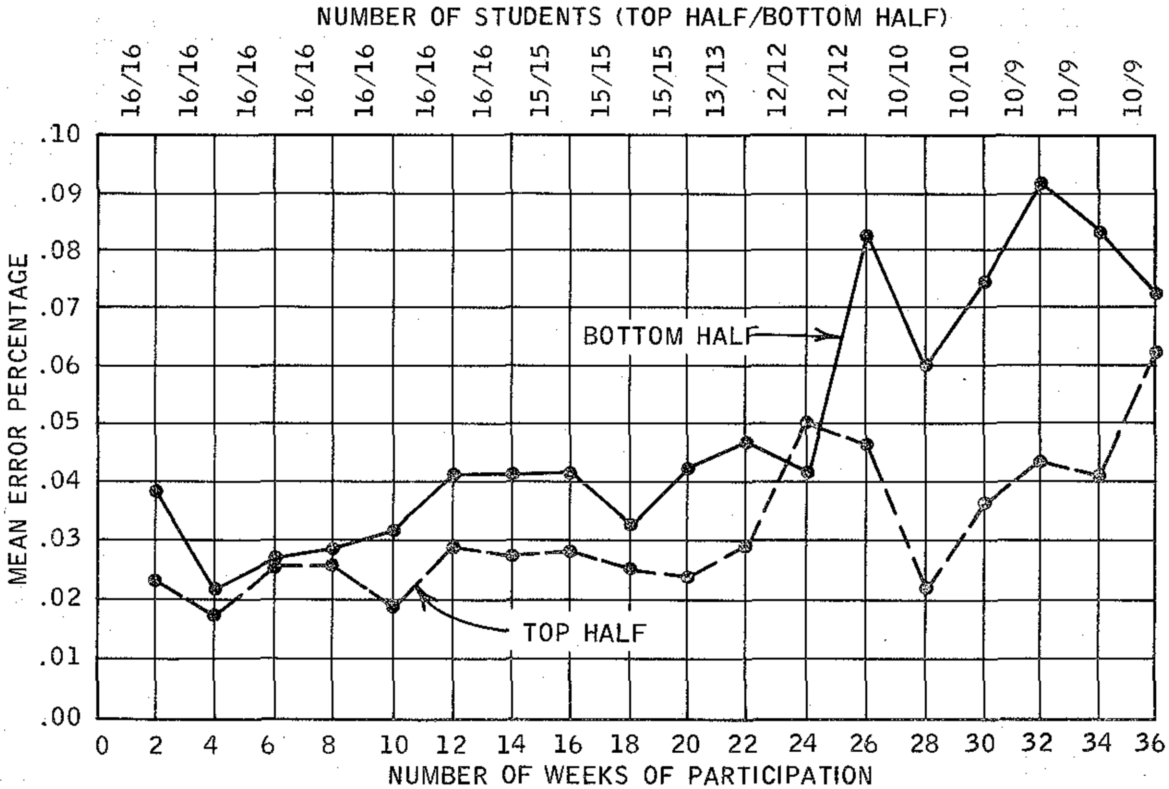

Figure 7. Mean percentage of problems completed which were in error.

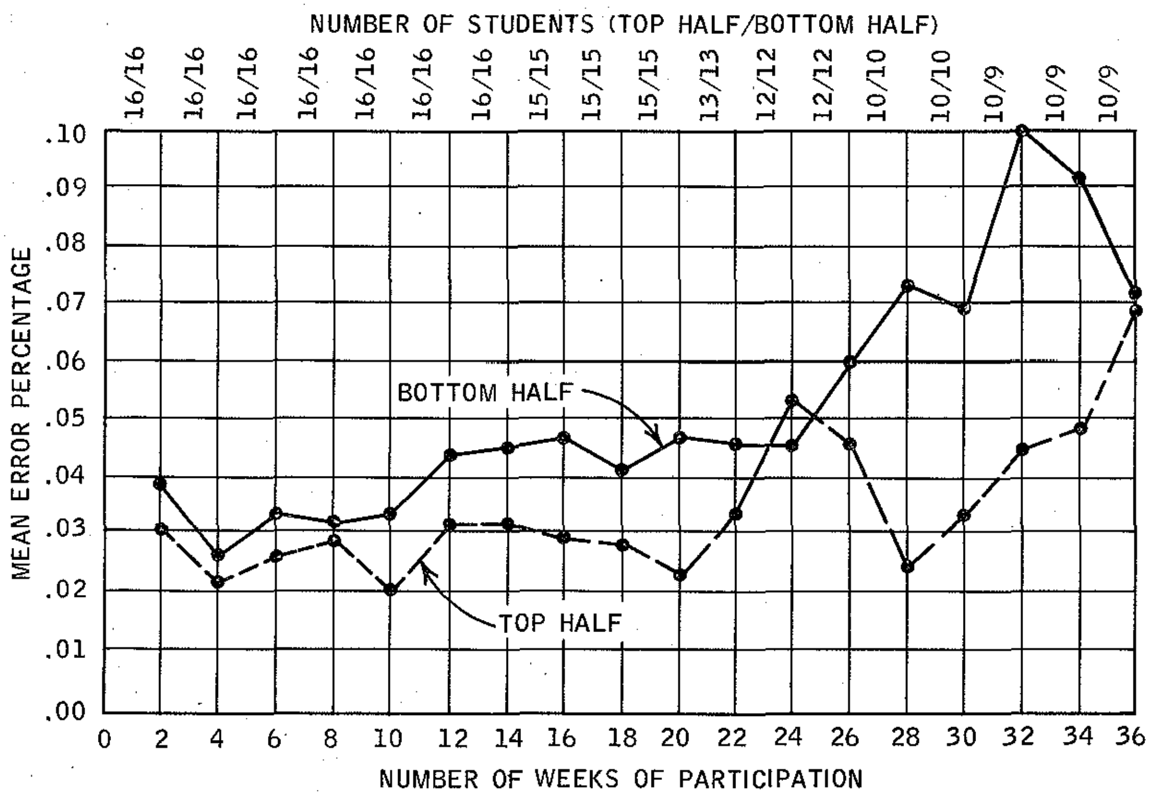

Figure 8. Mean percentage of responses completed which were in error.

for bright second and third graders. This is one of the reasons we have introduced an increasingly larger body of supplementary material as the students become more deeply immersed in the mathematics curriculum of the upper elementary grades.

The achievement tests administered this year to the students again show a high level of performance. Results of the Sets and Numbers tests, administered after completion of the appropriate section, are given in Table 4. There was slightly more individual variation this year, espe-

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Insert Table 4 about here
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cially for Book 3B, Part 1. The inter-subject variation for the Greater Cleveland Mathematics Tests was again very low, as is clear from the data shown in Table 4, and the evidence of high achievement quite apparent.

During the summer session the students were given at least one set of practice-and-maintenance drills each day. The problems were displayed on the CRT's These drills mainly reviewed basic arithmetical skills as covered in Sets and Numbers, Books 2A, 2B, 3A, and 3B. The mean results are shown in Table 5. This table gives mean error results as well

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Insert Table 5 about here
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as average total elapsed time for each drill. (A drill as referred to in the table consists of from 5 to 20 exercises depending on the problem type.) Without going into a more elaborate analysis it is apparent that the error rate on several of the drills is higher than desirable, and we were not satisfied with the mastery of basic arithmetic skills as reflected in these results. Consequently during the academic year 1965-66, we are making a more concerted effort to provide continual practice on basic skills by giving a short, five- or six-minute drill almost every day, even though the main curriculum work consists of extensive supplementary material.

This report emphasizes the mean quantitative results for the group of students, but a great deal of qualitative anecdotal material of considerable interest is in the permanent file of the project, and will at least be summarized in a later report. In this and the previous annual

Table 4. Mathematics Achievement Tests

Test	Group Mean	Standard Deviation	Total Possible Score	N
Sets and Numbers				
Book 2A--Part I	37.9	1.7	40	32
Book 2A--Part II	30.2	4.6	34	32
Book 2B--Part I	25.9	3.9	30	32
Book 2B--Part II	25.7	2.8	28	31
Book 3A--Part I	38.1	5.2	44	31
Book 3A--Part II	32.1	6.9	40	28
Book 3B--Part I	31.9	15.1	39	17
Book 3B--Part II	36.5	3.4	40	12
Book 4A--Part I	46.4	4.5	55	7
Book 4A--Part II	18.4	1.3	20	5
Book 4A--Part III	34.3	5.6	40	4
Book 4A--Part IV	30.0	none	32	1
Book 4A--Part V	27.0	none	27	1
Book 4A--Part VI	30.0	none	30	1
Book 4A--Part VII	34.0	none	37	1
Greater Cleveland Mathematics Tests for Grade 2				
1	28.0	1.2	29	32
2	28.8	2.0	31	32
3	26.1	3.7	31	32
4	27.1	3.0	31	32

Table 5. Summer-Session Drill Results

Drill	Average Number of Total Errors	Average Probability Wrong	Average Elapsed Time in Seconds
1	.7	.02	173
2	3.3	.11	273
3	2.4	.08	252
4	4.2	.14	290
5	3.2	.11	262
6	4.2	.14	262
7	8.9	.30	378
8	7.4	.25	382
9	6.9	.23	369
10	7.9	.26	335
11	5.5	.18	296
12	2.9	.59	174
13	7.0	.23	429
14	5.7	.19	313
15	6.3	.21	307
16	5.3	.18	284
17	9.1	.30	396
18	8.6	.29	357
19	12.3	.41	426
20	6.1	.20	309
21	15.0	.50	544
22	12.4	.41	490
23	6.6	.22	361
24	9.6	.32	389
25	11.6	.39	501
26	5.8	.16	310
27	9.1	.30	382

report an emphasis has been placed on quantitative results, because of the almost total absence of such records in the published literature.

Study of the personality and social development of the students is being continued under the direction of Professor Pauline S. Sears, and the results will be published elsewhere.