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APPROXIMATE PROBABILITY AND EXPECTATION OF GAMBLES

I. INTRODUCTION

After many years of work in the theory of measurement and closely related topics in the foundations of probability, I have come to feel that a sustained attack on the theory of approximate measurement is needed. There are two main reasons for this conviction. The first is that actual scientific procedures of measurement have such an approximate character and there is consequently an obvious need to capture in explicit axiomatic form the implicit assumptions underlying actual procedures.

The second reason is of more general philosophical interest. The historical tendency since the work of Simpson, Lagrange, and Laplace in the eighteenth century is to treat the approximate character of measurement procedures as due to *errors* of measurement arising either from 'systematic' or 'random' effects. This concept of error matches up with Laplacean determinism to give weight to a theory of exact results of measurement being in principle, if not in practice, possible. The fairy-tale quality of this view of measurement in physics should be evident enough two hundred years after its first telling. Even if it is not, I shall not attempt to make it evident here, but restrict myself to the more special arena of personal beliefs and decisions and their measurement.

It is surely surprising if not literally paradoxical that the view of ideal measurement characteristic of classical determinism has been carried over almost untouched to modern theories of partial belief and decision. It is, for example, a characteristic requirement of Savage's (1954) theory of decisions that the axioms on preferences between decisions be strong enough to lead to a numerical representation in terms of a *unique* probability distribution on beliefs and a utility function on consequences unique up to a linear transformation.

As in the case of classical physics there is a mythology of ideal measurement leading to unique numerical results which is widespread in contemporary theories of belief and decision. But the most casual common-

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sense consideration of actual beliefs and decisions generates almost immediate skepticism about any numerical uniqueness claim. The desire for uniqueness I view as a remnant of past hopes for an orderly, thoroughly deterministic universe.

In a previous paper (Suppes, 1974), I developed a simple theory for approximate measurement of beliefs, with the representation in terms of upper and lower probabilities. The objective of the present paper is to extend the theory to the simplest case of decisions, namely, gambles. In this restricted case the axioms have an especially simple form. In a forthcoming paper (Suppes, 1975) the theory is broadened to cover general decisions.

Some difficulties that arise in extending the theory to a complete representation in terms of upper and lower expectations are also explicitly discussed.

II. PRIMITIVE CONCEPTS

The axioms are based on five primitive concepts. To begin with, there is the set X of possible outcomes, and an algebra $\mathfrak{F}$ of events which are subsets of X. The algebra $\mathfrak{F}$ is a nonempty family of subsets of X closed under complementation and union. The third concept is the subalgebra $\mathscr{S}$ of standard events. The finite number of standard events that make up $\mathscr{S}$ correspond to standard weights in the case of the measurement of mass. From a practical standpoint, $\mathscr{S}$ can be realized by flipping what is believed to be a fair coin n times, where n is fixed in advance. The choice of n corresponds to choosing in physics a desired precision of measurement. Exact measurement results are obtained for the elements of $\mathscr{S}$, and other measurements are approximated in terms of standard events, which is again very similar to much measurement practice in the physical sciences and engineering.

The fourth primitive concept is the set U of consequences. Essentially no structural restrictions are placed on U, not even implicit cardinality restrictions, except that U be nonempty. It is useful to remark that there is a similar absence of cardinality restrictions on X, the set of possible outcomes. This absence of cardinality restrictions which seems to violate Archimedean requirements of measurement will be commented on later.

The fifth primitive concept is the central one of binary preference between gambles. Technically the relation $\succcurlyeq$ is a binary relation on

$U \times \mathfrak{F} \times U$. A gamble is written xAy, and is interpreted as follows: if event A occurs, x is the consequence or payoff, and if event not A occurs, y is the payoff. Thus, the notation $xAy \succcurlyeq uBv$ means that gamble xAy is weakly preferred to gamble uBv.

The relation of strict preference $\succ$ is defined in the usual manner in terms of weak preference, as is the relation $\approx$ of indifference. If α and β are gambles, then

$$\alpha \succ \beta \quad \text{if and only if} \quad \alpha \succcurlyeq \beta \quad \text{and not} \quad \beta \succcurlyeq \alpha,$$
$$\alpha \approx \beta \quad \text{if and only if} \quad \alpha \succcurlyeq \beta \quad \text{and} \quad \beta \succcurlyeq \alpha.$$

It is also convenient to define a binary relation of preference just on U:

$$x \succcurlyeq y \quad \text{if and only if} \quad xXy \succcurlyeq yXx.$$

Note that gamble xXy has x as the certain payoff and gamble yXx has y as the certain payoff.

It is convenient in stating the axioms to refer to $U \times \mathfrak{F} \times U$ as the set of all gambles. Moreover, if S is in $\mathscr{S}$ then xSy is a *standard gamble*, or a *gamble with standard event.*

Finally, as is customary, $\emptyset$ is the empty set, i.e., the impossible event, so that gamble $x\emptyset y$ has y as the certain outcome or payoff.

III. AXIOMS

The seven axioms, all elementary in character, are embodied in the following definition.

DEFINITION 1. *A structure* $\langle X, \mathfrak{F}, \mathscr{S}, U, \succcurlyeq \rangle$ *is an approximate structure for gambles if and only if the following axioms are satisfied for any* A, B, *and* C *in* $\mathfrak{F}$, *any* S *and* S' *in* $\mathscr{S}$ *and any* x, y, u, *and* v *in* U:

Axiom 1. [$\mathscr{S}$-*finiteness*] *The set* $\mathscr{S}$ *of standard events is finite;*

Axiom 2. [*Order*] *The relation* $\succcurlyeq$ *weak orders the set of gambles;*

Axiom 3. [*Non-negative probabilities*] *If* $x \succcurlyeq y$ *then* $xAy \succcurlyeq x\emptyset y$;

Axiom 4. [*Monotonicity*] *If* $x \succcurlyeq y$ *and* $xAy \succcurlyeq xBy$, *then if* $u \succcurlyeq v$ *then* $uAv \succcurlyeq uBv$;

Axiom 5. [*Additivity*] *If* $A \cap C = B \cap C = \emptyset$ *and* $x \succcurlyeq y$ *then* $xAy \succcurlyeq xBy$ *if and only if* $xA \cup Cy \succcurlyeq xB \cup Cy$;

Axiom 6. [$\mathscr{S}$*-positivity*] *There is an x and y in U such that for every $S \neq \emptyset$ in $\mathscr{S}$, $xSy \succ x\emptyset y$;*

Axiom 7. [$\mathscr{S}$*-solvability*] *If $xSy \succcurlyeq xS'y$ and $x \succcurlyeq y$ then there is an S'' in $\mathscr{S}$ such that $xSy \approx xS' \cup S''y$.*

The meaning of each of the axioms should be essentially transparent. In Axiom 2 the assertion that the relation $\succcurlyeq$ weak orders the set of gambles is equivalent to requiring that the relation $\succcurlyeq$ be transitive and strongly connected on this set.

There is no Archimedean axiom on the set $\mathfrak{F}$ of general events, but this is because the role of this axiom is covered by Axioms 1 and 7, i.e., by the finiteness and solvability of the set $\mathscr{S}$ of standard events.

The intuitive idea back of Axiom 4 is this. If $x \succcurlyeq y$ and $xAy \succcurlyeq xBy$ then we expect A to be at least as probable as y, but this probabilistic relation should not depend on x and y, and thus the monotonicity requirement that it also hold for any u and v such that $u \succcurlyeq v$.

IV. REPRESENTATION THEOREMS

The first theorem represents subjective probabilities or beliefs in terms of upper and lower probabilities. The three basic properties of an upper probability measure P^* on an algebra of sets and the corresponding lower measure P_* are these:

(I) $\quad P_*(X) = P^*(X) = 1$;

(II) $\quad P_*(\emptyset) = 0$;

(III) $\quad$ If $A \cap B = \emptyset$ then $P_*(A) + P_*(B) \leqslant P_*(A \cup B) \leqslant P_*(A) + P^*(B) \leqslant P^*(A \cup B) \leqslant P^*(A) + P^*(B)$.

For present purposes it will be useful to cite from Suppes (1974) a qualitative characterization of approximate probability and the theorem leading to the existence of an upper and lower probability measure. The basic representation theorem for probability can then be proved by deriving the qualitative probability axioms given below as consequences of the axioms for gambling decisions given above.

DEFINITION 2. *A structure $\mathscr{X} = \langle X, \mathfrak{F}, \mathscr{S}, \succcurlyeq \rangle$ is a finite approximate measurement structure for beliefs if and only if X is a nonempty set,*

$\mathfrak{F}$ and $\mathcal{S}$ are algebras of sets on X, and the following axioms are satisfied for every A, B, and C in $\mathfrak{F}$ and every S and T in $\mathcal{S}$:

Axiom P1. *$\mathcal{S}$ is a finite subset of $\mathfrak{F}$;*

Axiom P2. *The relation $\succcurlyeq$ is a weak ordering of $\mathfrak{F}$;*

Axiom P3. *$A \succcurlyeq \emptyset$;*

Axiom P4. *If $A \cap C = \emptyset$ and $B \cap C = \emptyset$ then $A \succcurlyeq B$ if and only if $A \cup C \succcurlyeq B \cup C$;*

Axiom P5. *If $S \neq \emptyset$ then $S \succ \emptyset$;*

Axiom P6. *If $S \succcurlyeq T$ then there is a V in $\mathcal{S}$ such that $S \approx T \cup V$.*

Note that Axioms 2, 3, and 4 are just the familiar de Finetti qualitative axioms. The additional de Finetti axiom that $X \succ \emptyset$ is an immediate consequence of Axiom 5, because X is a standard event.

The basic theorem for finite approximate measurement structures for beliefs is then the following. (The proof is given in Suppes, 1974.)

In the formulation of the theorem it is useful to refer to *minimal elements* of $\mathcal{S}$, i.e., standard events that are atoms of the subalgebra $\mathcal{S}$. It is also useful to refer to *minimal intervals* (S, S') of standard events. An interval (S, S') is minimal if $S \prec S'$ and there is a minimal event S'' (or atom) of $\mathcal{S}$ such that $S \cup S'' \approx S'$.

THEOREM 1. *Let $\mathcal{X} = \langle X, \mathfrak{F}, \mathcal{S}, \succcurlyeq \rangle$ be a finite approximate measurement structure for beliefs. Then*

(i) *there exists a probability measure P on $\mathcal{S}$ such that for any two standard events S and T*

$$S \succcurlyeq T \quad \text{if and only if} \quad P(S) \geqslant P(T),$$

(ii) *the measure P is unique and assigns the same positive probability to each minimal event of $\mathcal{S}$,*

(iii) *if we define P_* and P^* as follows:*

(a) *for any event A in $\mathfrak{F}$ order-equivalent to some standard event S,*

$$P_*(A) = P^*(A) = P(S),$$

(b) *for any A in $\mathfrak{F}$ not order-equivalent to some standard event S, but lying in the minimal open interval (S, S') for standard events S and S'*

$$P_*(A) = P(S) \quad \text{and} \quad P^*(A) = P(S'),$$

then P_* *and* P^* *satisfy conditions* (I – III) *for upper and lower probabilities on* $\mathfrak{F}$, *and*

(c) *if n is the number of minimal elements in* $\mathscr{S}$ *then for every A in* $\mathfrak{F}$

$$P^*(A) - P_*(A) \leqslant 1/n.$$

The representation theorem for approximate probabilities for gambles depends on Theorem 1. We also need to define for gambling structures the corresponding order relation of qualitative probability: for A and B in $\mathfrak{F}$, $A \succcurlyeq B$ if and only if for every x and y in U if $x \succcurlyeq y$ then $xAy \succcurlyeq xBy$.

THEOREM 2. *Let* $\langle X, \mathfrak{F}, \mathscr{S}, U, \succcurlyeq \rangle$ *be an approximate structure for gambles. Then there is a unique probability measure on* $\mathscr{S}$ *and upper and lower probabilities on* $\mathfrak{F}$ *satisfying conditions* (i)–(iii) *of Theorem* 1.

Proof. As already indicated, the approach is to show that events ordered by $\succcurlyeq$ satisfy the axioms given in Definition 2 and thereby (i)–(iii) are satisfied immediately, in accordance with Theorem 1. To distinguish the axioms in question (those of Definition 1), I have numbered them P1–P6.

The first, P1, is just Axiom 1 (of Definition 1) which asserts that the set of standard events is finite.

The second, P2, asserts that the relation $\succcurlyeq$ weakly orders $\mathfrak{F}$. Let us first consider transitivity. By hypothesis $A \succcurlyeq B$ and $B \succcurlyeq C$, and thus by definition

$$(\forall x, y)\ (\text{if}\quad x \succcurlyeq y \quad \text{then} \quad xAy \succcurlyeq xBy)$$
$$(\forall x, y)\ (\text{if}\quad x \succcurlyeq y \quad \text{then} \quad xBy \succcurlyeq xCy).$$

Assume now

(1) $\qquad x \succcurlyeq y;$

then we have at once $xAy \succcurlyeq xBy$ and $xBy \succcurlyeq xCy$, whence by transitivity of $\succcurlyeq$ on $U \times \mathfrak{F} \times U$, we have $xAy \succcurlyeq xCy$, and since this is true for any x and y satisfying (1), we have by definition $A \succcurlyeq C$, as desired.

To complete the proof of P2 we must establish the strong connectivity of $\succcurlyeq$ on $\mathfrak{F}$. By virtue of Axiom 2, we have

$$xAy \succcurlyeq xBy \quad \text{or} \quad xBy \succcurlyeq xAy.$$

We need consider only the case when $x \succcurlyeq y$. Suppose

$$xAy \succcurlyeq xBy,$$

then by the monotonicity axiom, Axiom 4, for every u and v

$$uAv \succcurlyeq uBv,$$

and hence

$$A \succcurlyeq B.$$

On the supposition that $xBy \succcurlyeq xAy$, we would by similar argument establish that $B \succcurlyeq A$, which completes the proof of strong connectedness.

Probability Axiom P3 asserts that for every event A, $A \succcurlyeq \emptyset$. This follows at once from Axioms 3 and 4.

Probability Axiom P4 asserts the 'additivity' of qualitative probabilities. Clearly proof of P4 depends most directly on Axiom 5, the additivity axiom for the present setup. The hypothesis that $A \cap C = B \cap C = \emptyset$ is common to both. Assume now that $A \succcurlyeq B$. Then by definition if $x \succcurlyeq y$ then $xAy \succcurlyeq xBy$, whence by Axiom 5, $xA \cup Cy \succcurlyeq xB \cup Cy$, and using monotonicity, we infer $A \cup C \succcurlyeq B \cup C$. By a similar argument from $A \cup C \succcurlyeq B \cup C$, we may infer $A \succcurlyeq B$, which establishes P4.

Probability Axiom P5 asserts that every nonempty standard event S is strictly positive in probability. Proof of this obviously depends on Axiom 6. Assume that $S \neq \emptyset$. Then by Axiom 6, there is an x_0 and a y_0 such that $x_0 S y_0 \succ x_0 \emptyset y_0$. Note first that $x_0 \succ y_0$, because $x_0 X y_0 \succ x_0 \emptyset y_0$ and $x_0 \emptyset y_0 \approx y_0 X x_0$. Thus by monotonicity $S \succ \emptyset$.

Finally, Probability Axiom P6 requires $\mathscr{S}$-solvability in the form that if $S \succcurlyeq S'$ then there is an S'' in $\mathscr{S}$ such that $S \approx S' \cup S''$, and this follows easily from Axiom 7 and monotonicity. Q.E.D.

Turning now to upper and lower expectations and utilities, we would like to have upper and lower utilities with the property that for every x in U

(IV) $$u_*(x) \leqslant u^*(x),$$

and, more importantly, similar to (III), we would like to have upper and lower expectations with the property:

(V) $$u_*(x)\,P_*(A) + u_*(y)\,P_*(\bar{A}) \leqslant E_*(xAy) \leqslant E^*(xAy) \leqslant u^*(x)\,P^*(A) + u^*(y)\,P^*(\bar{A}).$$

Properties (I)–(III) are classical in measure theory since the beginning of the century, although they have not been used in axiomatic studies of fundamental measurement until recently. Properties (IV) and (V) are essentially new, but represent obvious generalizations to utility and expectation of the approximate character of upper and lower probability measures.

Unfortunately, Property (V) cannot be satisfied by models of Definition 1. Explicit consideration of a counterexample will make it easier to explain the underlying reason for the failure. We may proceed in the following way. Let us fix x_0 and y_0 and let

$$u(x_0) = u_*(x_0) = u^*(x_0) = 1$$
$$u(y_0) = u_*(y_0) = u^*(y_0) = 0,$$

and furthermore, we consider only consequences x such that $x_0 \succcurlyeq x \succcurlyeq y_0$. Then we have an 'exact' theory of the expectations $E(x_0 S y_0)$ for all standard events S, namely,

$$E(x_0 S y_0) = u(x_0) P(S) + u(y_0) P(\bar{S}).$$

Now we define

$$u_*(x) = E(x_0 S y_0)$$
$$u^*(x) = E(x_0 S' y_0),$$

where (S', S) is a minimal interval with $x_0 S' y_0 \succ x \succ x_0 S y_0$. Similar definitions apply for upper and lower expectations:

$$E_*(xAy) = E(x_0 S y_0)$$
$$E^*(xAy) = E(x_0 S' y_0),$$

with $x_0 S' y_0 \succ xAy \succ x_0 S y_0$. Of course, if equivalence holds in any of these cases, e.g., if $xAy \approx x_0 S y_0$, then the upper and lower measures are the same.

In this context, we may construct a simple counterexample to Property (V) by taking as the algebra of standard events $\mathscr{S} = \{X, S, \bar{S}, \emptyset\}$, and thus $P(S) = P(\bar{S}) = \frac{1}{2}$. We need only consider upper measures to illustrate the difficulties. Let $u^*(x) = \frac{1}{2}$ and $P^*(A) = P^*(\bar{A}) = \frac{1}{2}$. Then clearly for our restricted $\mathscr{S}$, we must have $E^*(xAy_0) = \frac{1}{2}$, but

$$u^*(x) P^*(A) + u^*(y_0) P^*(\bar{A}) = \tfrac{1}{4},$$

and this violates Property (V).

The source of the difficulty is the existence of the multiplicative terms $u^*(x) P^*(A)$ and $u^*(y) P^*(\bar{A})$. In the case of the upper and lower probability measures above these multiplicative terms do not arise, and Property (III) gives rise only to subadditive and superadditive conditions, which can be satisfied by the finite standard event construction.

What can be established for structures satisfying Definition 1 is a numerical representation in terms of a semiorder, defined along the lines indicated above. Keeping x_0 and y_0 fixed as above, but not necessarily requiring that x_0 and y_0 be bounds, i.e., that $x_0 \succcurlyeq x \succcurlyeq y_0$, we define $*\succ$ as follows:

$$xAy *\succ uBv \text{ if and only if there is a standard event } S \text{ such that } xAy \succ x_0 S y_0 \succ uBv.$$

It is easy to show that the relation $*\succ$ has the property of being a semiorder, i.e., satisfies the following axioms for all gambles α, β, γ, and δ:

Axiom S1. *Not* $\alpha *\succ \alpha$;

Axiom S2. *If* $\alpha *\succ \beta$ *and* $\gamma *\succ \delta$ *then* $\alpha *\succ \delta$ *or* $\gamma *\succ \beta$;

Axiom S3. *If* $\alpha *\succ \beta$ *and* $\beta *\succ \gamma$ *then* $\alpha *\succ \delta$ *or* $\delta *\succ \gamma$.

We then have the following theorem.

THEOREM 3. *Let* $\langle X, \mathfrak{F}, \mathscr{S}, U, \succcurlyeq \rangle$ *be an approximate structure for gambles. Then*

(i) $*\succ$ *is a semiorder on the set of all gambles;*

(ii) $xAy *\succ uBv$ *if and only if* $E^*(xAy) \geqslant E^*(uBv) + 1/n$, *where n is the number of atoms in* $\mathscr{S}$;

(iii) $xAy *\succ uBv$ *if and only if* $E_*(xAy) \geqslant E_*(uBv) + 1/n$.

The proof of Theorem 3 is straightforward and is therefore omitted. The numerical representations (ii) and (iii) are worth noting, because no such numerical representation in terms of the weak ordering $\succcurlyeq$ is possible in the absence of cardinality restrictions on $\mathfrak{F}$ and U. The uniform threshold $1/n$ of the semiorder arises from the finiteness of $\mathscr{S}$; the representation given is a significant one for infinite semiorders that have such a finite subset, independent of the present context of decisions and gambles.

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