BEHAVIORISTIC FOUNDATIONS OF UTILITY

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BEHAVIORISTIC FOUNDATIONS OF UTILITY[1]

Patrick Suppes

In the past two decades there has been an intensive development of the subject of decision making. A variety of objectives and viewpoints has dominated the constructive as well as the critical work on the subject. Nonetheless a pervasive goal of nearly all contributors has been the elucidation of a theory of rationality for purposive behavior in situations of risk and uncertainty. Intuitively we expect every considered judgment or decision of a serious person to be rational in some definite sense. Certain authorities would maintain even that every considered decision of any mammalian organism is rational in the sense of representing the attempt to maximize some significant quantity. The most prominent "maximization" analysis of rationality is the thesis that the decision maker should maximize expected utility or value with respect to his beliefs concerning the facts of the situation. To perform this maximization, he needs to have, or to act as if he had, a subjective probability function measuring his degrees of belief and a utility function measuring the relative value to him of the various possible outcomes of his actions or decisions.

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1 This research was supported by the Rockefeller Foundation and the Group Psychology Branch of the Office of Naval Research. I have benefited from conversations with several people on the topic of this paper, but most particularly from those with Donald Davidson, William K. Estes and Duncan Luce. Portions of this paper were presented at an International Colloquium on decision theory in Paris on May 27, 1959.

It is not my purpose here to expound the expected utility theory of behavior. An excellent detailed and leisurely analysis is Savage [13]. Rather my concern is to explore the extent to which behavioristic foundations can be supplied for utility. And I am using the term 'behavioristic' in the rather narrow sense of the experimental psychologist. The static character of the concepts of subjective probability and utility is suspect to the psychologist and he resists accepting them as basic concepts of behavior. Ideally, what is desired is a dynamic theory of the inherent or environmental factors determining the acquisition of a particular set of beliefs or values. Moreover, in the notions of stimulus, response and reinforcement the experimental psychologist has a triad of concepts which have proved adequate to explain much simple choice behavior. It is, therefore, a scientific problem of some interest to try to use just these behavioristic notions to derive a theory of subjective probability and utility.

In the first section I set forth the fundamental assumptions of stimulus sampling learning theory, which is the most formally sophisticated theory yet stated in terms of the concepts of stimulus, response and reinforcement. In the second section I attempt to show how this theory may be used to derive a utility function for various simple choice situations. This derived utility function is for stochastic choice behavior of the kind studied by Davidson and Marschak [3], Luce [10], Papandreou [12] and others. In the third and final section the earlier results are related to Shannon's concept of entropy and Luce's choice axiom.

1. STIMULUS SAMPLING LEARNING THEORY

The basic theory to be used in this paper is a modification of stimulus sampling theory as first formulated by Estes and Burke [2], [8], [9]. It is most closely connected with a formulation given by Suppes and Atkinson [17], but it also differs, in ways indicated below, from the latter. The axioms are formulated verbally here, and although there is no attempt in this paper to give a mathematically exact statement of the theory, it is hoped that the relation between the fundamental axioms and the results derived later will be reasonably clear, even to the reader without prior familiarity with the literature.

The first group of axioms deals with the conditioning of sampled stimuli, the second group with the sampling of stimuli, and the third with responses.

CONDITIONING AXIOMS

C1. *On every trial each stimulus element is conditioned to exactly one response.*

C2. *If a stimulus element is sampled on a trial it becomes conditioned with probability* θ *to the response (if any) which is reinforced on that trial.*

C3. *If no reinforcement occurs on a trial there is a probability that the sampled stimulus becomes conditioned to some other response.*

C4. *Stimulus elements which are not sampled on a given trial do not change their conditioning on that trial.*

C5. *The probability of a sampled stimulus element being conditioned is independent of the trial number and the outcome of preceding trials.*

SAMPLING AXIOMS

S1. _Exactly one stimulus element is sampled on each trial._

S2. _If on a given trial it is known what stimuli are available for sampling, then no further knowledge of the subject's past behavior or of the past pattern of reinforcement will change the probability of sampling a given element._

RESPONSE AXIOM

R1. _On any trial that response is made to which the sampled stimulus element is conditioned._

Detailed remarks about these axioms are to be found in Suppes and Atkinson [17]. The major change from the version in [17] is to be found in Axiom C3. There this axiom reads: "If no reinforcement occurs on a trial there is no change in conditioning on that trial." For the kind of experimental situation to be considered below it is natural to adopt the modified axiom given above as C3. A slight change in Axiom C5 has been made to accommodate the major change in C3; otherwise the axioms given here are those of [17].

Many readers may be particularly critical of the first sampling axiom, S1. There are at least two different kinds of remarks to be made in defense of the assumption that exactly one stimulus element is sampled on each trial. In the first place, this assumption is mathematically extremely convenient and it is scarcely possible to distinguish, for the kind of experiments to be described here, between it and more "liberal" sampling axioms, as for example the assumption that all stimulus elements in the basic stimulus set are sampled with independent probabilities. Secondly, S1 may be made more intuitively plausible by interpreting 'stimulus element' to mean <u>pattern of stimuli</u>, for it may be maintained that in any given situation an organism, at any given moment, is sampling exactly one pattern of stimuli. (For a more detailed discussion of the pattern concept, see Estes [7].)

We may consider two simple applications, which will be integrated into our discussion of utility in the next section. These two examples should serve adequately to illustrate how the basic axioms of stimulus sampling theory are related to particular experimental situations in order to make predictions about response behavior.

Suppose the task presented a subject is to predict on each trial exactly which one of two lights will come on. Thus on each trial exactly one of two reinforcing events, E_1 or E_2, occurs. The subject indicates his prediction at the beginning of each trial by pressing one of two keys, response A_1 or A_2, where A_i is the key under light E_i. The sequence of events on a given trial may be described thus:

trial begins with stimuli conditioned to A_1 or A_2 $\rightarrow$ stimulus sampled $\rightarrow$ response A_1 or A_2 $\rightarrow$ reinforcement E_1 or E_2 $\rightarrow$ possible change in conditioning of sampled stimulus .

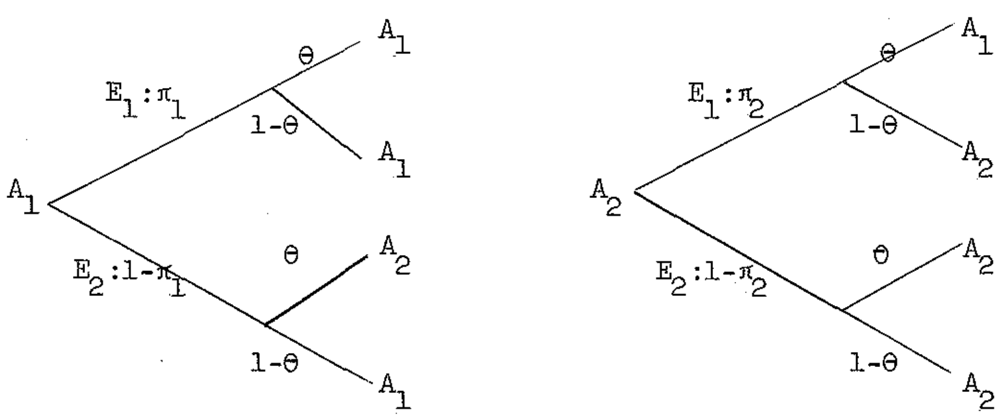
Using the "independence of path" assumptions represented by Axioms C5 and S2, it may be shown that if we assume that the stimulus set S consists of exactly one element then the sequence of response random variables

$< \underline{A}_1, \underline{A}_2, \ldots, \underline{A}_n, \ldots >$ is a Markov chain for many schedules of reinforcement satisfying the experimental conditions just described. (Here the value for each n of the random variable $\underline{A}_n$ is 1 or 2, according to whether the A_1 or A_2 response is made on trial n.) Using this result about Markov chains and the description of events on a trial, we may, upon imposition of a particular schedule of reinforcement, derive the transition matrix of the Markov chain. For consideration at this point we introduce the <u>simple contingent</u> case of reinforcement, namely, the probability of an E_1 or E_2 reinforcement on trial n depends only on the response made on trial n. Thus, using notation common in the literature:

$$P(E_1 \mid A_1) = \pi_1 ,$$

$$P(E_1 \mid A_2) = \pi_2 .$$

The states of the Markov process are A_1 and A_2. Being in state A_1, for instance, means that the single stimulus element is conditioned to A_1. The trees of the process are then:



The probabilities θ and $1-\theta$ occurring in the final branches of the trees are derived from Axiom C2, which is concerned with the conditioning of stimulus elements. For example, in the lower half of the first tree, an E_2 reinforcement occurs with probability $1-\pi_1$ after the initial response A_1. This initial response means that the single stimulus element is connected (or conditioned) to A_1. However, an E_2 reinforcement occurs. With probability θ this reinforcement is effective in changing the connection or conditioning of the single stimulus element to the A_2 response.

We immediately derive from the two trees the following transition matrix for the Markov chain:

	A_1	A_2
A_1	$1-\theta(1-\pi_1)$	$\theta(1-\pi_1)$
A_2	$\theta\pi_2$	$1-\theta\pi_2$

The asymptotic probability p_∞ of an A_1 response is easily computed from this matrix. The probability p_{n+1} of being in state A_1 is just:

$$p_{n+1} = p_{11}p_n + p_{21}(1-p_n) \ ,$$

where p_{ij} is the transition probability of going from A_i to A_j in one trial. (Thus p_{ij} is just the entry for the i^{th} row and j^{th} column of the transition matrix.) Now at asymptote

$$p_{n+1} = p_n = p_\infty \ ,$$

whence

$$p_\infty = (1 - \theta(1 - \pi_1))p_\infty + \theta\,\pi_2(1 - p_\infty) \ ,$$

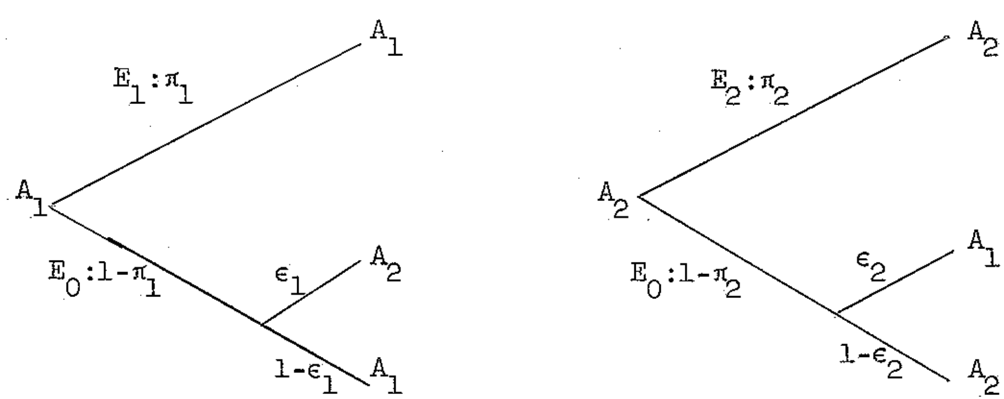
and this simple linear equation has as its solution

$$p_\infty = \frac{\pi_2}{1 - \pi_1 + \pi_2} \ . \qquad (1)$$

It is worth noting that the asymptotic probability p_∞ is independent of the conditioning parameter θ . Experimental evidence supporting equation (1) is to be found in Estes [6].

Rather than derive further predictions for the simple contingent case of reinforcement, I now turn to the second example, which I shall call the two-arm bandit case of reinforcement. The name stems from the resemblance of the experimental situation to that of playing a slot machine with two arms or levers rather than one; on each trial a choice between the levers is made. (Mathematical statisticians have, during the past few years, considered in detail what is the optimal way to play a two-arm bandit for a finite number of trials when the probabilities of pay-off of the two arms are unknown.)

The experimental situation, then, consists of choosing on each trial between two levers. In the experiment to be described in somewhat more detail in the next section, lever 1 is given a probability π_1 of paying, and lever 2 a probability π_2 . Unlike the simple contingent case there is no "correction" procedure, i.e., the subject is not told, or led to believe, that on each trial exactly one of the arms of the "bandit" will pay off. If he chooses lever 1, say, then either it pays off or it does not, without reference to the possible choice of lever 2. Such an analysis of reinforcement leads to an application of Axiom C3: if lever i is chosen (i.e., response A_i occurs) and no reward or reinforcement follows (event E_0 occurs), then there is a probability ϵ_i that the sampled stimulus will become conditioned to the other response, i.e., choosing the other lever. Application of C3 to the present situation seems natural and intuitively sound, but it is to be emphasized that any uniform method, applicable to many other experiments, for handling nonreinforcement trials would be premature in view of the highly conflicting experimental evidence obtained by various investigators, particularly in connection with the extinction of learning. The trees for the one-element model may be drawn as follows (we have eliminated the θ and $1-\theta$ branches in case of reward, for they lead to the same result, namely, retention in the same state with which the trial began):



(Note that we use E_0 to designate the event of no reinforcement.)

The trees yield as the transition matrix of the Markov chain:

	A_1	A_2
A_1	$1-\epsilon_1(1-\pi_1)$	$\epsilon_1(1-\pi_1)$
A_2	$\epsilon_2(1-\pi_2)$	$1-\epsilon_2(1-\pi_2)$

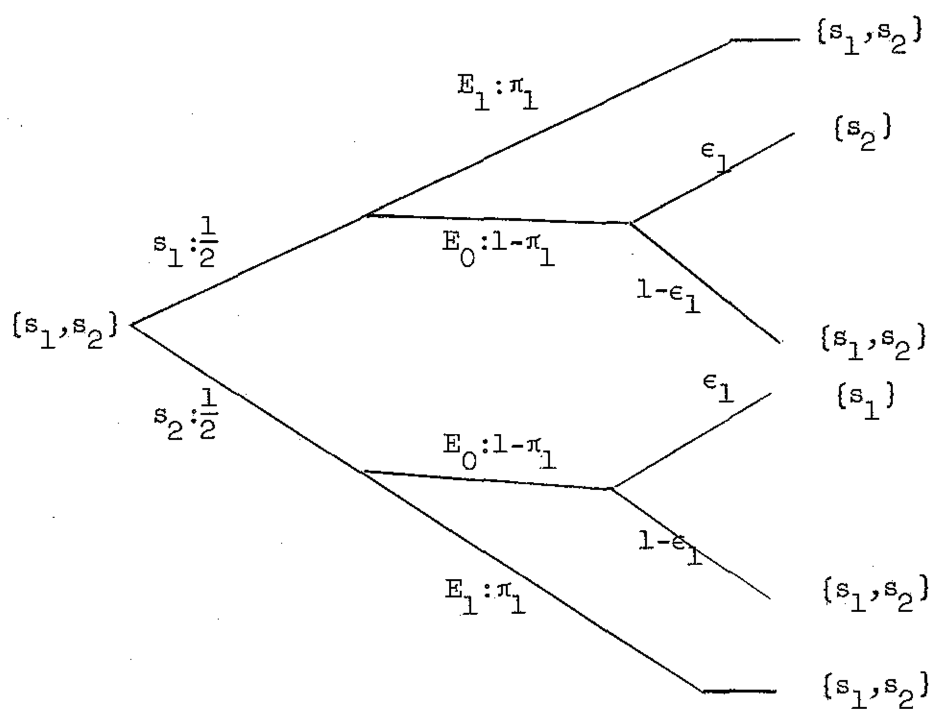
And by the same line of argument which led to equation (1) we obtain as the asymptotic probability p_∞ of the A_1 response for the two-arm bandit:

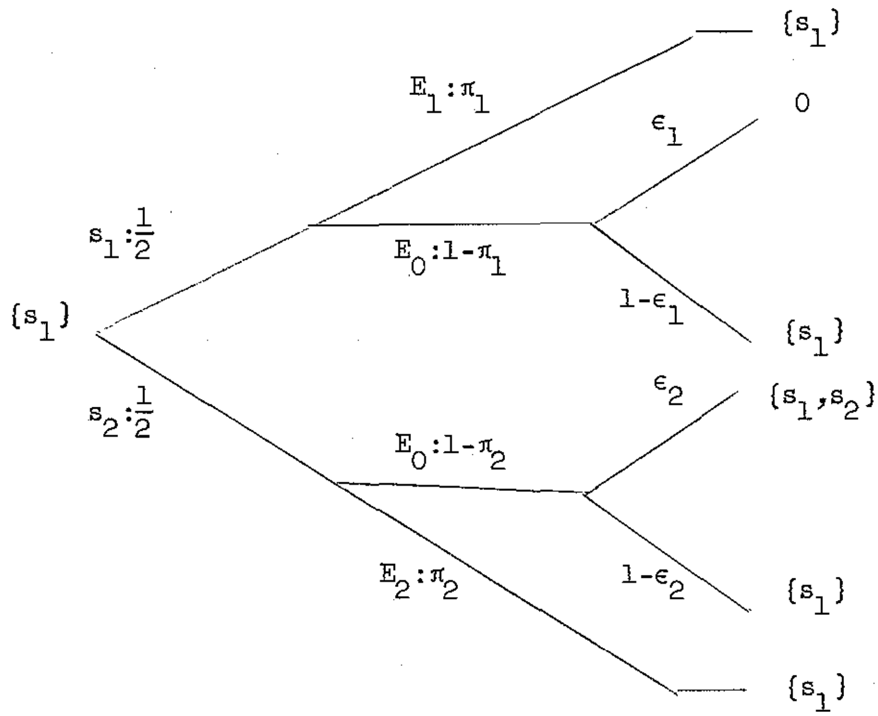
(2) $$p_\infty = \frac{\epsilon_2(1-\pi_2)}{\epsilon_1(1-\pi_1)+\epsilon_2(1-\pi_2)} \quad .$$

If $\epsilon_1 = \epsilon_2$, equation (2) simplifies to:

(3) $$p_\infty = \frac{1-\pi_2}{(1-\pi_1)+(1-\pi_2)} \quad .$$

In connection with these two applications of stimulus sampling theory, it is important to emphasize that the asymptotic probabilities (1) and (2) do not in any way depend on the assumption that there is exactly one stimulus element. In fact, the results (1) and (2) hold on the assumption of any finite number of stimulus elements. To illustrate the methods of working with more than one stimulus element, we may write down some of the trees and the transition matrix for the two-element model as applied to the case of the two-arm bandit. The states of the Markov chain are no longer the responses A_1 and A_2 , but the possible partitions representing the conditioning of the two stimulus elements. Let s_1 and s_2 be the two elements. We may indicate any partition of the set $\{s_1,s_2\}$ between the two responses A_1 and A_2 simply by indicating which elements are conditioned to A_1. Thus the four states of the process may be denoted by $\{s_1,s_2\}$, $\{s_1\}$, $\{s_2\}$ and 0, where 0 is the empty set (meaning here that neither s_1 nor s_2 is conditioned to A_1 if the subject is in state 0). We give the trees when the subject begins in either state $\{s_1,s_2\}$ or $\{s_1\}$; the other two trees are similar to these. The one assumption needed, and not given in our fundamental axioms, is the probability of sampling s_1 as against that of sampling s_2. Here we assume there is an equal chance of sampling either, although this is not very crucial to any of our results.





Note that in the first tree either E_1 or E_0 must occur since both stimulus elements are conditioned to A_1, and thus only the A_1 response occurs regardless of which element is sampled. This is not the case for the second tree; if s_1 is sampled A_1 occurs and then either E_1 or E_0, but if s_2 is sampled A_2 occurs and then either E_2 or E_0. The transition matrix to be derived from these two trees and the other two not shown here is the following:

	$\{s_1,s_2\}$	$\{s_1\}$	$\{s_2\}$	0
$\{s_1,s_2\}$	$1-\epsilon_1(1-\pi_1)$	$\frac{1}{2}\epsilon_1(1-\pi_1)$	$\frac{1}{2}\epsilon_1(1-\pi_1)$	0
$\{s_1\}$	$\frac{1}{2}\epsilon_2(1-\pi_2)$	$1-\frac{1}{2}\epsilon_1(1-\pi_1)-\frac{1}{2}\epsilon_2(1-\pi_2)$	0	$\frac{1}{2}\epsilon_1(1-\pi_1)$
$\{s_2\}$	$\frac{1}{2}\epsilon_2(1-\pi_2)$	0	$1-\frac{1}{2}\epsilon_1(1-\pi_1)-\frac{1}{2}\epsilon_2(1-\pi_2)$	$\frac{1}{2}\epsilon_1(1-\pi_1)$
0	0	$\frac{1}{2}\epsilon_2(1-\pi_2)$	$\frac{1}{2}\epsilon_2(1-\pi_2)$	$1-\epsilon_2(1-\pi_2)$

Note that the probability of an A_1 response when in state $\{s_1,s_2\}$ is one, when in states $\{s_1\}$ or $\{s_2\}$ is $\frac{1}{2}$, and when in state 0 is zero. Whence from computation of the asymptotic probabilities for each state we may at once determine the asymptotic probability of an A_1 response. As already remarked, the result is again equation (2). We shall not consider the details of these computations here. In fact, at this point we end the consideration of stimulus sampling theory in order to turn to utility theory proper.

2. UTILITY

As indicated in the introductory section, in this paper I am mainly concerned with a utility function for the kind of choice behavior which has come to be labeled, not entirely happily, "stochastic." Roughly speaking, the central character of stochastic choice behavior is that upon presentation of two alternatives a and b, with a choice of one required, under essentially identical circumstances sometimes a will be chosen by a subject and sometimes b. Let $p(a,b)$, then, be the probability that a is chosen over b . A (<u>stochastic</u>) <u>utility</u> <u>function</u> for a set of alternatives A is a real-valued function u defined on A such that for every a, b, c and d in A

(4) $$p(a,b) \geq p(c,d) \text{ if and only if } u(a) - u(b) \geq u(c) - u(d) .$$

Combining results in Suppes and Winet [18], Suppes [16] and Davidson and Marschak [3], it may be shown that if the set A and the probabilities p(a,b) satisfy the following axioms, then there exists a stochastic utility function for A , and moreover this function is unique up to a positive linear transformation.

Axiom U1. $p(a,b) + p(b,a) = 1$.

Axiom U2. $0 < p(a,b) < 1$.

Axiom U3. *If* $p(a,b) \geq p(c,d)$ *then* $p(a,c) \geq p(b,d)$.

Axiom U4. *There is a* c *in* A *such that*

$$p(a,c) = p(c,b) .$$

Axiom U5. *If* $p(c,d) > p(a,b) > \frac{1}{2}$ *then there is an* e *in* A *such that* $p(c,e) > \frac{1}{2}$ *and* $p(e,d) \geq p(a,b)$.

Axiom U6. (*Archimedean Axiom*). *If* $p(a,b) > \frac{1}{2}$ *then for every probability* q *such that* $p(a,b) > q > \frac{1}{2}$ *there is a positive integer* n *such that* $q \geq p(a,c_1) = p(c_1,c_2) = \ldots = p(c_n,b) > \frac{1}{2}$.

Now one implication of these six axioms is that A must be an infinite set if for at least two members a and b of A , $p(a,b) \neq \frac{1}{2}$. Simple and natural conditions, which are not unduly restricted and which will guarantee existence of a stochastic utility function for a finite set A , are not easily found. An unworkable recursive, but not finite, axiomatization can be given by enumerating for each n all isomorphism types. Some of the fundamental difficulties of finite axiomatization are brought out in Scott and Suppes [14]. The upshot of these axiomatic problems, it seems to me, is that for finite sets of alternatives we have no clear and intuitively natural ideas in terms only of probabilities of choice of the notion of utility, and thus of the notion of rationality for such situations.[2]

On the other hand, we may apply the results of the preceding section to indicate how from the axioms of stimulus sampling theory a utility function may be derived for finite sets of alternatives. To begin with, let us consider

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2 Under a rather natural continuity assumption, which is however stronger than U4 - U6, Debreu [5] has shown that the quadruple condition (U3) is necessary and sufficient for the existence of a utility function satisfying (4). Of course, granted U4 - U6, and the "technical axioms" U1 and U2, it is obvious that the quadruple condition is also necessary and sufficient in this context. It may also be remarked that to give necessary and sufficient conditions on the set A and the function p , without continuity or finiteness restrictions, is the extremely difficult mathematical problem of classifying all isomorphism types representable by a real-valued function u satisfying (4).

the second example of the application of stimulus sampling theory, namely, the two-arm bandit. On each trial the subject must choose between two alternatives, but now, to make the utility considerations interesting, we assume there is a set of alternatives available, with choice restricted on each trial to one of a pair. Clearly alternative a does not in and of itself have more value than alternative b ; the value of a is determined by the probability of pay-off, as is that of b . Thus the experimenter may manipulate the value of any alternative according to his determination of its pay-off function. We seek a function u satisfying (4). Now according to (2) of the last section, at asymptote,

$$p(a,b) = \frac{\epsilon_b(1-\pi_b)}{\epsilon_a(1-\pi_a)+\epsilon_b(1-\pi_b)} \tag{5}$$

where π_a is the probability of pay-off of alternative a when it is chosen, ϵ_a is the probability the sampled stimulus will become conditioned to the other alternative when the choice of a is not rewarded, and similar definitions hold for π_b and ϵ_b . In view of (5) to satisfy (4), we need to find a function u such that

$$\frac{\epsilon_b(1-\pi_b)}{\epsilon_a(1-\pi_a)+\epsilon_b(1-\pi_b)} \geq \frac{\epsilon_d(1-\pi_d)}{\epsilon_c(1-\pi_c)+\epsilon_d(1-\pi_d)} \text{ if and only if } u(a)-u(b) \geq u(c)-u(d). \tag{6}$$

Let $\rho_a = \epsilon_a(1-\pi_a)$ for every a in A .[3] The right-hand inequality of (6) may then be written:

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3 I assume throughout that $0 < \pi_a, \epsilon_a < 1$, for every a in A .

(7) $$\frac{\rho_b}{\rho_a + \rho_b} \geq \frac{\rho_d}{\rho_c + \rho_d} ,$$

but (7) holds, if and only if

$$\rho_b \rho_c \geq \rho_a \rho_d ,$$

which holds, if and only if

$$\frac{\rho_b}{\rho_a} \geq \frac{\rho_d}{\rho_c} ,$$

which again holds, if and only if

$$\frac{\frac{1}{\rho_a}}{\frac{1}{\rho_b}} \geq \frac{\frac{1}{\rho_c}}{\frac{1}{\rho_c}} ,$$

which, finally, holds, if and only if

(8) $$\log \frac{1}{\rho_a} - \log \frac{1}{\rho_b} \geq \log \frac{1}{\rho_c} - \log \frac{1}{\rho_d} .$$

From (6), (7) and (8) we conclude that an appropriate utility function is, for a in the set A of alternatives:

(9) $$u(a) = \log \frac{1}{\epsilon_a (1 - \pi_a)} .$$

If $\epsilon_a = \epsilon_b$ for every a and b in A , we may take the simpler function

$$u'(a) = \log \frac{1}{1 - \pi_a} \quad .$$

It is straightforward to show that the utility function defined by (9) is unique up to a positive linear transformation if the reasonable restriction is made that any acceptable utility function must be continuous in ϵ_a and π_a . Moreover, from the existence of a function u satisfying (4), it immediately follows that the asymptotic choice behavior predicted by stimulus sampling theory satisfies all the various conditions of weak and strong stochastic transitivity discussed in the literature, as well as the quadruple condition expressed by Axiom U3 above. It should be mentioned that these results do not necessarily hold during the course of learning; in particular the utility function defined by (9) does not satisfy (4) during the course of learning. This fact, it seems to me, accords well with the widespread assumption, albeit often tacit, that the utility function of a person is an equilibrium concept.

It is, of course, to be emphasized that the utility function defined by (9) is not that of the mathematical statistician bent on maximizing his monetary pay-off in the long run. It should be abundantly clear that the whole theory of probabilistic choice behavior is not meant to apply to such a person. For under the pay-off conditions defined here, if $\pi_a > \pi_b$ the statistician should have at asymptote $p(a,b) = 1$. The point of (9) is rather to define a utility function which may be used to predict the actual behavior of all but the statistically sophisticated few. Numerous empirical studies (Mosteller and Nogee [11], Davidson, Suppes and Siegel [4], Papandreou [12], Atkinson and Suppes [1], Davidson and Marschak [3]) have clearly shown that naive subjects do not behave like mathematical statisticians. Experimental data on utility functions as defined by (9) for the two-arm bandit situation will be reported elsewhere.

The preceding analysis also has direct application to the first example of simple contingent reinforcement discussed in the preceding section. By replacing π_2 by $1-\pi_2$, for purposes of symmetry, thus having as reinforcement probabilities $P(E_1 \mid A_1) = \pi_1$ and $P(E_2 \mid A_2) = \pi_2$, we may, obviously, get a utility function satisfying (4) by taking

$$u(a) = \log \frac{1}{1-\pi_a} \quad .$$

Further remarks on this case do not seem necessary.

The interesting question of generalization, it seems to me, is that of considering situations in which choice is made from one of n alternatives. In classical economic theory, the resolution of this choice problem is immediate: simply choose the most preferred item. But, as far as I know, with the notable exception of Luce [10] there has been little if any analysis of stochastic choice behavior when the choice set has more than two alternatives. To describe this situation, let us use the notation $p(a,A)$ to mean the probability a is chosen in preference to any member of A, with the understanding that $\{a\} \cup A$ is the full choice set available, i.e.,

$p(a,A) + p(A,a) = 1$, where $p(A,a)$ means the probability an element of A is chosen in preference to a .[4] Beginning simply with $p(a,A)$, it is far from clear to me what axioms of rational behavior one might expect an organism to satisfy, in order to guarantee the existence of a utility function. In fact, it is not completely obvious what should be the defining characteristic of a utility function. In analogy to (4) I suggest:

(10) $$p(a,A) \geq p(b,B) \text{ if and only if } u(a) - u(A) \geq u(b) - u(B) .$$

Condition (10) requires the utility of a set of alternatives to be defined, but it by no means implies that this set function need be additive, i.e., we need not have if A and B are disjoint sets that

$$u(A \cup B) = u(A) + u(B) \quad .$$

On the other hand, the intuitive interpretation of $p(a,A)$ suggests that if A is a subset of B then the utility of A is equal to or less than B , for in some sense the utility of A is the overall value weighting assigned to the set in deciding to choose a rather than any member of A . Also, it seems reasonable to require that if the utility of A is equal to or greater than that of B and a set C is added to both A and B , with C disjoint from both A and B , then the utility of $A \cup C$ is equal to or greater than that of $B \cup C$. These two principles may be summarized:

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4 From this point on, X rather than A will represent the total set of available alternatives.

(11) <u>if</u> $A \subseteq B$ <u>then</u> $u(A) \leq u(B)$,

(12) <u>if</u> $A \cap C = B \cap C = 0$ <u>and</u> $u(A) \leq u(B)$ <u>then</u> $u(A \cup C) \leq u(B \cup C)$.

(Evidently (11) and (12) would not be acceptable if some of the alternatives had negative pay-offs, a possibility which we exclude here.)

What I now want to show is that for this multi-choice case a utility function satisfying (10), (11) and (12) may be derived from the axioms of stimulus sampling theory by generalizing the approach to the two-arm bandit problem. For simplicity I shall again consider only the model with one stimulus element, although the results given here may easily be extended to a finite number of stimulus elements. The axioms given in the preceding section do need to be supplemented in one important respect, namely, we shall make Axiom C_3 more definite by assuming that when a chosen response is not reinforced, the probability of the stimulus element becoming conditioned to some other response is uniformly distributed over the remaining set of available responses. Thus, in the notation of Section 2, if there are n other available responses and total probability ϵ_i that the stimulus element will become conditioned to some other response than A_i after A_i is not reinforced, then ϵ_i/n is the probability it will become conditioned to A_j , for $j \neq i$ and A_j in the available set. Keeping this notation in mind, it is easy to see that the transition matrix for $n+1$ possible responses (i.e., $n+1$ possible choices) has the following form:

(13)

	A_1	A_2	. . .	A_{n+1}
A_1	$1-\epsilon_1(1-\pi_1)$	$\frac{\epsilon_1}{n}(1-\pi_1)$	. . .	$\frac{\epsilon_1}{n}(1-\pi_1)$
A_2	$\frac{\epsilon_2}{n}(1-\pi_2)$	$1-\epsilon_2(1-\pi_2)$	. . .	$\frac{\epsilon_2}{n}(1-\pi_2)$
⋮	. . .	. . .	. . .	. . .
A_{n+1}	$\frac{\epsilon_{n+1}}{n}(1-\pi_{n+1})$	$\frac{\epsilon_{n+1}}{n}(1-\pi_{n+1})$	. . .	$1-\epsilon_{n+1}(1-\pi_{n+1})$

Following standard notation, let u_j be the asymptotic probability of response A_j . Then, as is well known, the asymptotic probabilities u_j may be obtained as the solution of the system of linear equations

$$(14)\quad \begin{cases} u_j = (1-\epsilon_j(1-\pi_j))u_j + \sum_{i\neq j} \frac{\epsilon_i(1-\pi_i)}{n} , \quad \text{for} \quad j = 1,\ldots, n+1 \\ \sum u_j = 1 , \end{cases}$$

provided the matrix (13) satisfies certain regularity conditions, which are indeed satisfied here because every entry in the matrix is strictly positive.

It is not difficult to show that the solution of (14) is:

$$(15)\qquad u_j = \frac{\prod_{i\neq j} \epsilon_i(1-\pi_i)}{\sum_j \prod_{i\neq j} \epsilon_i(1-\pi_i)} .$$

Now $p(a,A) = u_a$, and if we divide the numerator and denominator of the right-hand side of (15) by $\prod_{j\in X} \rho_j$, where as before $\rho_j = 1/\epsilon_j(1 - \pi_j)$ and the set of alternatives is $X = A \cup \{a\}$, then

(16) $$p(a,A) = \frac{1/\rho_a}{\sum_{j\in X} 1/\rho_j} .$$

On the basis of (16) we have a simple chain of equivalences like that leading from (7) to (8), which yields that $p(a,A) \geq p(b,B)$ if and only if

(17) $$\log 1/\rho_a - \log \sum_{j\in A} 1/\rho_j \geq \log 1/\rho_b - \log \sum_{j\in B} 1/\rho_j ,$$

and thus to satisfy (10), we define a utility function u for any non-empty finite set A of alternatives as:

(18) $$u(A) = \log \sum_{j\in A} 1/\rho_j .$$

Moreover, we may use (18) to generalize (10) immediately to the probabilities $p(A,\tilde{A})$, where $\tilde{A}$ is the complement of the set A with respect to the total set of alternatives, i.e., $A \cup \tilde{A} = X$. The interpretation of $p(A,\tilde{A})$ is that this is the probability of choosing an alternative from A rather than from its complement $\tilde{A}$. We observe first that (16) yields:

$$p(A,\tilde{A}) = \frac{\sum_{A} 1/\rho_j}{\sum_{A} 1/\rho_j + \sum_{\tilde{A}} 1/\rho_j} \quad . \tag{19}$$

Manipulations similar to those already carried out then result in:

(20) $p(A,\tilde{A}) \geq p(B,\tilde{B})$ if and only if $u(A) - u(\tilde{A}) \geq u(B) - u(\tilde{B})$.

It is easily verified that the utility function u defined by (18) satisfies (11) and (12) as well as (10) and (20). If u were also an additive set function it would be more appropriate to call it a subjective probability function. It seems to me that its logarithmic rather than additive character is intuitively sound. In particular, the marginal utility of adding another alternative to a set of such is appropriately a decreasing function of the size of the set. In other words, the utility function defined by (18) has the classical property that as wealth increases each additional unit has decreasing marginal utility.

4. RELATIONS TO OTHER THEORIES

To begin with, I want to show that the entropy of any set of alternatives X , probability distribution p , and partition π of X is a negative linear transformation of the expected utility of (X,p,π) .[5] Following the

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5 A partition of a set X is a family of non-empty, pairwise disjoint subsets of X such that the union of all sets in the family is X .

well-known work of Shannon (see, e.g., Shannon and Weaver [1949]) on the theory of information, the entropy H of (X,p,Π) is defined as:

$$H(\Pi) = -\sum_{A\in\Pi} p(A,\tilde{A})\log_2 p(A,\tilde{A}) \quad . \tag{20}$$

And the expected utility $\mathcal{E}(u,\Pi)$ is defined in the standard manner as:

$$\mathcal{E}(u,\Pi) = \sum_{A\in\Pi} p(A,\tilde{A})u(A) \quad . \tag{21}$$

Now

$$\begin{aligned} u(A) &= \log \sum_{A} 1/\rho_j \\ &= \log \frac{\sum_{A} 1/\rho_j}{\sum_{X} 1/\rho_j} + \log \sum_{X} 1/\rho_j \\ &= \alpha \log_2 p(A,\tilde{A}) + \beta \ , \end{aligned}$$

where $\alpha = \log 2$ and $\beta = \log \sum_{X} 1/\rho_j$, and it is clear α and β are both independent of Π .[6]

---

6 When no base of a logarithm is indicated, it is understood to be e .

Substituting this last result for u(A) into (21) we have

$$\mathcal{E}(u,\Pi) = \sum_{A\in\Pi} p(A,\tilde{A})[\alpha \log_2 p(A,\tilde{A}) + \beta]$$

$$= -\alpha H(\Pi) + \beta ,$$

the desired conclusion. It is to be noticed that the finest partition of X maximizes entropy, whereas the coursest one maximizes expected utility (with respect to the set of all partitions of X).

I now turn to consideration of Luce's choice axiom ([10], p. 6) which we may formulate as follows: if $A \subseteq B \subseteq X$ then

$$p_X(A) = p_B(A) p_X(B) , \tag{22}$$

where $p_X(A)$ is the probability that an element of A is selected from the total choice set X . Thus if $A \cup \tilde{A} = X$, then in the notation used earlier, $p_X(A) = p(A,\tilde{A})$. The purpose of the subscript usage is to indicate an explicit change in the total set of available alternatives.

Without further assumptions (22) cannot be derived from the postulates for learning theory given at the beginning, because they include no assertions about the constancy or continuity of behavior when the number of available responses is changed. However, to derive (22) we need add only the postulate that the conditioning parameter ϵ_i of response A_i for every i is independent of what subset of the alternatives X is available. Granted this additional assumption about conditioning, derivation of Luce's axiom is a simple matter, for

$$p_X(A) = \frac{\sum_A 1/\rho_j}{\sum_X 1/\rho_j}$$

$$= \frac{\sum_A 1/\rho_j}{\sum_B 1/\rho_j} \cdot \frac{\sum_B 1/\rho_j}{\sum_X 1/\rho_j}$$

$$= p_B(A)p_X(B) \quad .$$

Using his choice axiom Luce proves the existence of a ratio scale $v(j)$ ([10], pp. 20-28) with the property that

$$p_X(A) = \frac{\sum_A v(j)}{\sum_X v(j)} \quad .$$

The relation of this additive ratio scale to the utility function u defined by (18) is simply

$$v(A) = ke^{u(A)} \quad ,$$

where k is a positive real number.

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