MODELS OF INDIVIDUAL TRAJECTORIES IN COMPUTER-ASSISTED INSTRUCTION FOR DEAF STUDENTS

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MODELS OF INDIVIDUAL TRAJECTORIES IN COMPUTER-ASSISTED INSTRUCTION FOR DEAF STUDENTS[1]

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In this report we present a new approach to evaluation of curriculum. Many of us who have been engaged in curriculum reform efforts have been dissatisfied with the wait-and-see approach required when classical evaluation of a new curriculum is used. We have in mind evaluation by comparing pretests and posttests, with an analysis of posttest grade-placement distributions as a function of pretest distribution and exposure in some form to the new curriculum.

In line with approaches used in other parts of science, it is natural to ask if a more predictive-control approach could be used and made an integral part of the curriculum to ensure greater benefits, especially for the disadvantaged or handicapped student. The approach discussed in this report is aimed precisely at this question. The strategy is to develop a theory of prediction for individual student progress through the curriculum, to use this predictive mechanism as a means of control by regulating the amount of time spent on the curriculum by a given student, and to thereby achieve set objectives for the grade-placement gains of a student. Such an approach also calls for individualization in the objectives of a course, for it is unrealistic to expect all students to make the same gains in the same amount of time, or to expect that the slowest students can cover as much material as the best students simply by spending additional time in a course. Consequently, even with a differential approach to the amount of time each student may spend in the curriculum, it is still not reasonable to impose a uniform concept of grade-placement gain on all students.

Another important feature of our approach to the prediction of student progress is to separate the global features of the curriculum (described in the next section) by a simple differential equation from the global individual parameters characteristic of the individual student. In many respects, the estimation of the global individual parameters corresponds to the fixing of boundary conditions in the solution of differential equations in physics. In our case the boundary conditions correspond to the characteristics of the individual student and the differential equation itself to the structure of the curriculum. We do not know if the differential equation that fits the structure of the elementary-mathematics computer-assisted instruction (CAI) curriculum developed at Stanford over a number of years will be the characteristic differential equation of other curriculums. The generality of the qualitative assumptions from which the differential equation is derived provides some grounds for optimism. Examination of individual student trajectories in other courses will be required to test this optimism. (In the remainder of this report we shall often talk about <u>student trajectories</u> rather than student progress in order to give the sense of a definite path as a function of time that we are predicting for the individual student.)

THEORY

As we have already emphasized, our analysis is aimed at the global performance of the student. The fact that we are considering only his global progress, and not his performance on individual exercises, makes it possible for us to state general axioms about information processing from which we may derive the basic stochastic differential equation that we believe is characteristic of many different curriculums, especially curriculums that are tightly articulated and organized in their development. Certainly this is a characteristic of the CAI mathematics drill-and-practice curriculum considered in this report.

In our axioms we speak of new pieces of information. We did not want to use the technical concept of a bit of information, for in many instances the new information introduced at a given point in the curriculum constitutes in a literal sense a number of bits. The axioms are formulated in such a way that very little about information is assumed. A deeper analysis would aim at providing more structure to the theory of information outlined in our axioms. At the global level considered here it does not seem necessary.

The first axiom deals with a student's rate of processing or sampling information in a course. The second axiom postulates what happens to the student's mean rate of processing information when a new piece of information is introduced. The third axiom deals with the basic assumption about the rate of introducing new information. The fourth axiom assumes that the student's current position in a course is closely related to the sum of information introduced up to this point, and the fifth axiom makes a similar assumption about his rate of progress in the course. (Readers not interested in the technical statement of the axioms and the derivation of the basic differential equation should skip to the end of this section.)

For statement of the axioms and later use, we define the following quantities:

$y(t)$ = position of student in the course, and at $t = 0$ we set $y = 0$ for present purposes, but later consider a translation;

$\dot{y}(t)$ = rate of progress through the course;

$A(t)$ = cumulative amount of information introduced in the course up to time t;

$\dot{A}(t)$ = rate of introduction of information in the course;

$s(t)$ = student's rate of processing or sampling information.

The five general axioms are formulated as follows.

Axiom 1. A student's mean rate $s(t)$ of processing or sampling information is directly proportional to the rate of introduction of information in a course and inversely proportional to the total amount of information introduced up to time t, i.e., $s(t)$ is proportional to $\dot{A}(t)/A(t)$.

Axiom 2. Upon introduction of a new piece of information a student's new mean rate of processing information is decreased by an amount equal to the product of his current rate and the difference of his current rate and his asymptotic rate, i.e., for a small interval of time h

$$s(t + h) = s(t) - [s(t) - s(\infty)]\, s(t) .$$

Axiom 3. The probability of a new piece of information being introduced for a given student at time t is independent of t and the previous introduction of information.

Axiom 4. The position of a student in a course is directly proportional to the total information introduced thus far in the course, i.e., $y(t)$ is proportional to $A(t)$.

Axiom 5. The rate of progress of a student in a course is directly proportional to the rate of introduction of information in the course, i.e., $\dot{y}(t)$ is proportional to $\dot{A}(t)$.

Of the five axioms, it is clear that Axiom 2 is the least satisfactory in form. It could be formulated this way. The decrease in rate of processing upon introduction of a new piece of information falls off quadratically in the rate of processing. What we do not like is the absence of a more fundamental qualitative characterization of the rate assumption expressed in this axiom. Although we have given some thought to a reformulation of Axiom 2, we have not been successful in finding a genuinely better alternative.

We are reasonably satisfied with the other four axioms and believe that they have a natural intuitive content that does not require explicit discussion.

We turn now to the derivation of the basic stochastic differential equation. We emphasize that the equation is stochastic; it is a mean stochastic equation and not a deterministic one. Although the basic assumptions of the theory expressed in the five axioms permit us to derive more details about the behavior of students than is expressed in the mean stochastic equation, we shall not look at additional details in this report.

By Axiom 3, the introduction of new information is a Poisson process, let us say with parameter λ. Thus by Axiom 2, with probability λh in a small time interval h:

$$s(t + h) = s(t) - [s(t) - s(\infty)]\, s(t) , \tag{1}$$

with probability $o(h)$ more than one piece of information is introduced, and with probability $1 - \lambda h - o(h)$:

$$s(t + h) = s(t) , \tag{2}$$

whence from (1) and (2), and setting $s(\infty) = 0$, which seems intuitively sound,

$$\frac{s(t + h) - s(t)}{h} = -\lambda s(t)^2 + \frac{o(h)}{h} .$$

Hence, as $h \to 0$, we obtain the differential equation

$$\dot{s}(t) = -\lambda s^2(t) ,$$

whose solution is

$$s(t) = \frac{1}{\lambda t + c_1} .$$

By Axiom 1

$$s(t) = \frac{k_1 \mathring{A}(t)}{A(t)} , \qquad k_1 > 0 ,$$

but by Axiom 4

$$y(t) = k_2 A(t) , \qquad k_2 > 0 ,$$

and by Axiom 5

$$\dot{y}(t) = k_3\dot{A}(t) , \qquad k_3 > 0 ,$$

whence, combining results,

$$\frac{\dot{y}(t)}{y(t)} = \frac{k_4}{\lambda t + c_1} , \qquad k_4 > 0 .$$

Integrating this last equation, we obtain

$$\ln y(t) = \frac{k}{\lambda} \ln (\lambda t + c_1) + \ln |b_1| ,$$

and so

$$y(t) = b_1(\lambda t + c_1)^{k_4/\lambda} .$$

Here if $t = 0$, $y(t) = 0$, and so $c_1 = 0$. Assuming the student has some knowledge, c, of the course at $t = 0$, we take as our final equation

$$y(t) = bt^k + c .$$

As already indicated, the parameters b, c, and k are meant to be estimated separately for each individual student.

METHOD

The Mathematics Strands Curriculum

Assessment of the pedagogical effectiveness of the Institute's elementary mathematics curriculum on achievement among hearing-impaired students was reported in Suppes, Fletcher, Zanotti, Lorton, and Searle (1973). The present assessment is based on the kind of highly individualized study of trajectories outlined in the introduction of this report.

We briefly describe the strands program. A more detailed description is to be found in the report just referred to, or in Suppes, Goldberg, Kanz, Searle, and Stauffer (1971).

The objectives of the strands program are (a) to provide supplementary individualized instruction in elementary mathematics at a level of difficulty appropriate to each student's level of achievement, (b) to allow acceleration in any concept area in which a student demonstrates proficiency, and to allow repeated drill and practice in areas of deficiency, and (c) to report a daily profile of each student's progress through the curriculum.

A strand is a series of exercises of the same logical type (e.g., horizontal addition, vertical subtraction, multiplication of fractions) arranged sequentially in equivalence classes according to their relative difficulty. The 14 strands in the program and the grade levels spanned by each strand are shown in Table 1. Each strand contains either five

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Insert Table 1 about here

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or ten equivalence classes per half year, with each class labeled in terms of grade-placement (GP) equivalent. Data collected during several years of the earlier drill-and-practice mathematics program at Stanford were used to arrange the equivalence classes in an increasing order of difficulty and to ensure that new skills (e.g., regrouping in subtraction) were introduced at the appropriate point.

In addition to ordering the equivalence classes within a strand, we had to determine how much emphasis to give each strand at a given grade level. To determine this emphasis, we divided the curriculum into 14 parts, each corresponding to a half year. A probability distribution was defined for the proportion of problems on each strand for each half year. The final proportions in terms of time and problems for each half year for each strand are shown in Table 2.

TABLE 1

Grade Level Spanned by Each Strand in

the Elementary Mathematics Program

Strand	Content	Grade level
NUM	Number concepts	1.0-7.9
HAD	Horizontal addition	1.0-3.9
HSU	Horizontal subtraction	1.0-3.4
VAD	Vertical addition	1.0-5.9
VSU	Vertical subtraction	1.5-5.9
EQN	Equations	1.5-7.9
MEA	Measurement	1.5-7.9
HMU	Horizontal multiplication	2.5-5.4
LAW	Laws of arithmetic	3.0-7.9
VMU	Vertical multiplication	3.5-7.9
DIV	Division	3.5-7.9
FRA	Fractions	3.5-7.9
DEC	Decimals	4.0-7.9
NEG	Negative numbers	6.0-7.9

Insert Table 2 about here

A student's progress through the strands structure is purely a function of his own performance and is independent of the performance of other students; in fact, his progress on a given strand is independent of his own performance on other strands. A scheme defining movement through a strand uses the pattern of correct and incorrect responses to insure a rate of movement that reflects performance.

Equipment

The central computer processor was the Institute's PDP-10 system located on the Stanford campus. On-line, real-time communication was maintained with the participating schools located in California, Florida, Oklahoma, Texas, and the District of Columbia by means of dedicated telephone lines.

The student terminals were KSR Model-33 teletypewriters. The teletypewriters communicate information to and from the central computer system at a rate of about ten characters per second. All of the elementary mathematics exercises were typed at the terminal under computer control, and keyboard responses were given by the students. The details of exercise format and student responses are described in Suppes, Jerman, and Brian (1968) and Suppes and Morningstar (1972).

Students

The students participating in this experiment were chosen from the entire population of students who were enrolled in one of three residential schools for the deaf in California, Florida, and Texas and who

TABLE 2

Proportion of Time and Proportion of Problems for Each Strand for Each Half Year

Strand		Half year													
		1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
NUM	PT	50	24	24	17	10	5	7	7	8	11	14	10	15	15
	PP	36	18	16	12	10	4	8	8	10	14	20	10	19	19
HAD	PT	26	21	21	9	14	9								
	PP	32	28	26	10	14	8								
HSU	PT	14	10	16	9	4									
	PP	18	14	16	10	4									
VAD	PT	10	10	9	19	19	7	8	2	3	1				
	PP	14	12	12	22	20	6	10	2	4	2				
VSU	PT		9	8	15	22	10	13	3	3	1				
	PP		12	12	18	20	8	10	2	4	2				
EQN	PT		17	12	16	17	14	17	7	5	7	7	8	15	15
	PP		10	10	12	16	12	20	8	8	12	12	10	19	19
MEA	PT		9	10	8	8	11	7	7	5	5	5	5	5	5
	PP		6	8	6	6	6	8	8	8	8	8	6	6	6
HMU	PT				7	3	8	5	3	2					
	PP				10	6	14	10	6	8					
LAW	PT					3	5	5	3	3	3	1	1	8	8
	PP					4	6	6	4	6	6	2	2	10	10
VMU	PT						10	5	14	6	8	7	5	8	8
	PP						14	6	16	8	4	4	2	4	4
DIV	PT						15	22	34	48	33	40	13	14	14
	PP						18	10	16	16	6	8	2	3	3
FRA	PT						6	4	15	13	20	17	18	10	10
	PP						4	4	24	22	32	32	26	10	10
DEC	PT							7	5	4	11	7	36	10	10
	PP							8	6	6	14	10	38	10	10
NEG	PT											2	4	15	15
	PP											4	4	19	19

Note.--PT = proportion of time; PP = proportion of problems.

were receiving daily CAI sessions in the elementary-mathematics strands curriculum through the Institute's computer system in 1971-72. The degree of hearing loss among the students was essentially that adopted for admission standards by the schools; generally this loss averages at least 60 decibels in the better ear. All of the students from this population whose average GP was between 2.0 and 5.9, who had received more than twenty mathematics strands sessions and who were not assigned to the evaluation study already reported in Suppes et al. (1973), participated as subjects in the experiment. Complete data were obtained for 297 of the 355 students who began the experiment.

The on-line collection of data for these students began on February 14, 1972 and ended on May 5, 1972. Proctors supervising students' use of computer terminals were encouraged to have students take more than one CAI session per day where feasible, and especially to increase the number of sessions taken by students whose GP calibrated by the strands curriculum was low. Proctors and teachers were further encouraged to set GP objectives for individual students and to encourage them to take an adequate number of sessions to meet these GP objectives in terms of the GP calibration built into the strands mathematics curriculum.

Measures of Achievement

Three measures of achievement were taken. First, the final GP at the end of the experiment on the mathematics strands curriculum was immediately available. Second, the modified on-line Stanford Achievement Test called MSAT, developed at the Institute and described in detail in Suppes et al. (1973), was administered. Both of these measures were obtained for 297 of the 355 students who began the experiment. In addition, the Stanford Achievement Test (SAT) was administered off-line by the participating schools. Results on the computation section are available for 206 students and on the concepts and applications sections for 107 students.

RESULTS

Descriptive Statistics

We first describe for the 297 students who completed the experiment their beginning GP position in the mathematics strands curriculum and then their ending position. At the beginning of the experiment the mean position was a GP of 3.41 with a standard deviation of .828 and a range of 2.09 to 6.00. At the end of the experimental period the mean GP was 4.07 with a standard deviation of .844 and a range of 2.40 to 7.33.

During the experimental period, the mean number of CAI sessions averaged across the 297 students was 51.98 with a standard deviation of 15.91 and a range running from 25 to 146.

External Measurements of Achievement

In Table 3 the results of linear regressions are shown, using the final strands-curriculum GP of each student as the independent variable

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Insert Table 3 about here

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and the various external measures as dependent variables. Table 3 also shows the correlations between the final GP position of the students and the scores on the MSAT, SAT computation, SAT concepts, and SAT applications

TABLE 3

Linear Regressions with Final Strands GP as Independent Variable and Various Standard Achievement Measures as Dependent Variables

Independent variable	~	Dependent variable	R	SEE	F-ratio	Slope	Intercept	N
Final GP	~	MSAT	.86	.65	861.82**	1.31	-1.30	297
Final GP	~	SAT Computation	.80	.91	359.60**	1.44	-1.81	206
Final GP	~	SAT Concepts	.66	.82	80.57**	.74	.53	107
Final GP	~	SAT Applications	.53	.78	41.48**	.51	1.19	107
MSAT	~	SAT Computation	.83	.85	438.34**	1.03	-.06	206

** Significant, $p < .01$ ($F_{.99}(1,120) = 6.85$).

as described earlier. In all cases the F ratios are significant at $p < .01$. The correlation of .86 between the MSAT and the final GP for the 297 students is about as high as one could expect in any experiment of this sort. The correlation of .80 for the 206 students who completed the SAT computation section is also high.

The regression equation, for example, for the SAT computation section can be used to provide quite good predictions for what may be expected from students on the SAT computation section, given their final GP position in the mathematics strands curriculum. Such a regression equation can be useful as a predictive device in deciding how much supplementary drill and practice a student needs to show a reasonable GP gain as measured by a standard achievement test. On the other hand, it is the basic theoretical thrust of the present report to show how this essentially empirical regression approach can be improved by developing a theoretical model for student trajectories that permit better extrapolation, especially nonlinear extrapolation, of the effects of an increased number of CAI sessions, in order to determine the consequences of additional CAI sessions. Note that this theoretical approach uses a linear regression equation for the purposes of predicting an external score on a standard achievement test, but the theory enters in terms of deciding how many sessions a student should have to reach an agreed-upon objective measured in terms of strands GP.

In a previous evaluation study on the use of the mathematics strands curriculum by hearing-impaired students, reported in Suppes et al. (1973), the simple correlation of the strands final position of the 312 students in that experiment and the MSAT scores was .762. The corresponding simple correlation coefficient, not the correlation obtained from the regression equation, in the case of the present experiment for the 297 students was .797. As would be expected, the correlation obtained from the regression equation, with one more parameter free to estimate, is higher than either of the simple correlations. What is worth noting about these figures, however, is that in all three cases the correlations are around .8.

A regression of a different sort was used in Experiment 1 for the MSAT GP. The regression was run with the posttreatment MSAT as the dependent variable and with two independent variables, the pretreatment MSAT GP for each student and the number of CAI sessions. In this case the multiple correlation from the regression was .811, which again is close to those just mentioned, but in this case only the number of CAI sessions directly entered the regression and the GP itself did not. Of course, introduction of the pretreatment MSAT GP corresponds to the introduction of what is known generally from the literature to be a powerful predictive variable, namely, the pretreatment GP of the student on some standard measure.

Finally we should mention that the F-ratios in Table 3 are sufficiently high to warrant the judgment that the regression equations not only have a good correlation with a significant F-ratio, but also that the F-ratio is adequate to justify the use of the regression equations for predictive purposes. They are, in all cases, significant beyond $p = .01$ and in fact satisfy the four-times-significance-level rule sometimes quoted as desirable for predictive purposes.

Tests of the Theory

We turn now to tests of the theory and concentrate on the general equation for individual student trajectories resulting from the solution of the basic stochastic differential equation, which was itself derived from some simple qualitative postulates about information processing. We emphasized earlier that we take the differential equation to be characteristic of the course, but the three individual parameters present in the final equation are in principle to be estimated for each student individually. Recall that the basic equation is:

$$y(t) = bt^k + c \ .$$

In estimating individual parameters and fitting individual curves to individual student data, we have used three basic measures to evaluate the fit of the theory. The first and most important is the mean standard error in predicting the observation points for each student. The second is the mean absolute residual, that is, the mean absolute difference in the predictive and observed observations for each student, and the third is the mean of the maximum residuals for each student. To be explicit, let o_{ij} be observation i for student j and t_{ij} be the corresponding theoretical prediction. Then the standard error of n_j predictions for student j is:

$$\text{Standard error} = \left[1/n_j \sum_{i=1}^{n_j} (o_{ij} - t_{ij})^2\right]^{1/2}$$

and the mean standard error for the sample population of students is just the mean of their standard errors. To obtain the mean absolute residual for the sample population, we just replace $(o_{ij} - t_{ij})^2$ by

$|o_{ij} - t_{ij}|$ and do not take the square root; finally, in the case of the mean of the maximum residuals, we first find for each j

$$m_j = \max_i |o_{ij} - t_{ij}| ,$$

and then take the mean of the m_j's, i.e.,

$$\frac{1}{n} \sum_{j=1}^{n} m_j .$$

Regarding the number of observations per student, we fitted the theoretical curve by using the session number on which the student moved .1 of a GP. The GP for each student averaged across the 14 strands was computed only to .1 of a GP, and thus the times of change in recorded GP were the significant observations to use in fitting the theoretical curves. These observations may be regarded as defining a step function for the student's progress. In these terms, we fitted the theoretical curve to the points of discontinuity (i.e., change) in the step function. The average number of such points per student was approximately 12.

Before turning to the presentation of numerical data, we want to give a sense of how extremely close the fits of the theoretical curves are to the observed points for individual students. In Figures 1 to 4 we have presented results for four students whose exponents k vary

Insert Figures 1-4 about here

over a wide range. In particular, for one student the k value is taken at the limit, that is, we use the equation $y = b \ln t + c$, and for another student the other extreme of $k = 1.00$. We include as the third student that student whose standard error was the largest, namely, .197.

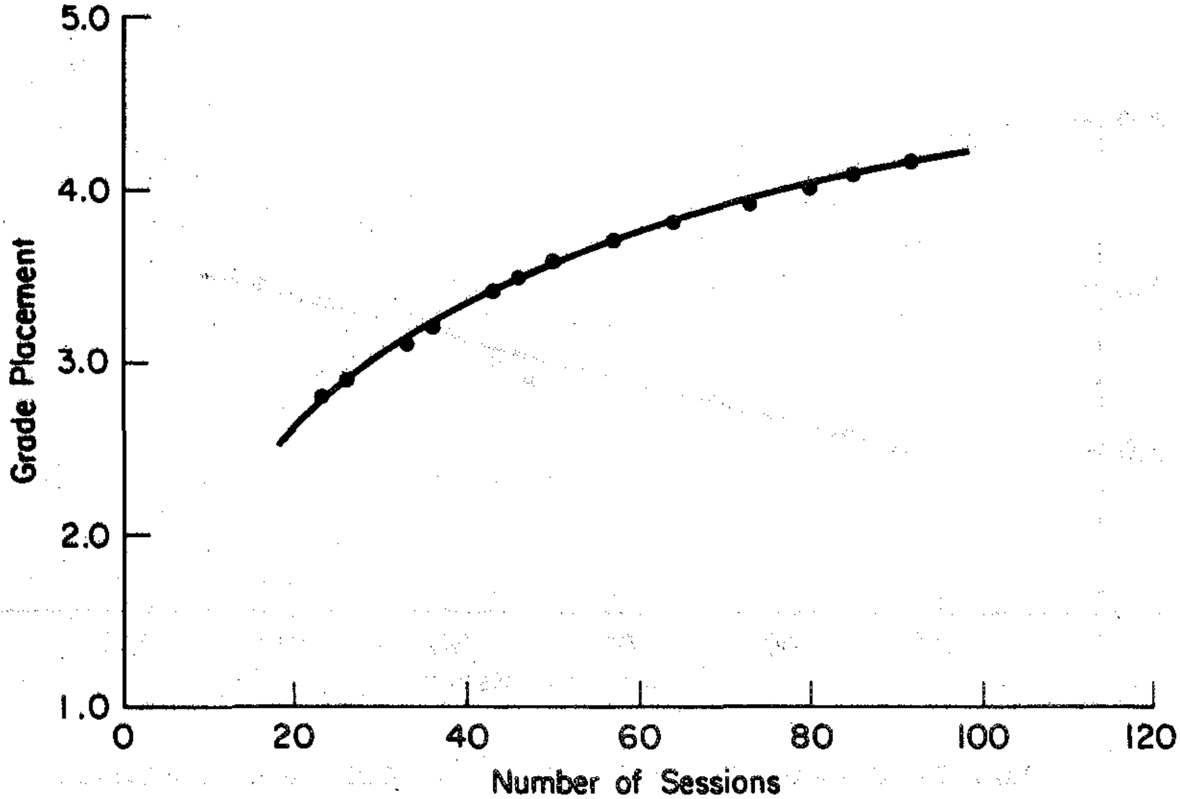

Fig. 1. Student with equation $y = b \ln t + c$, with $b = 1.00$ and $c = -.34$.

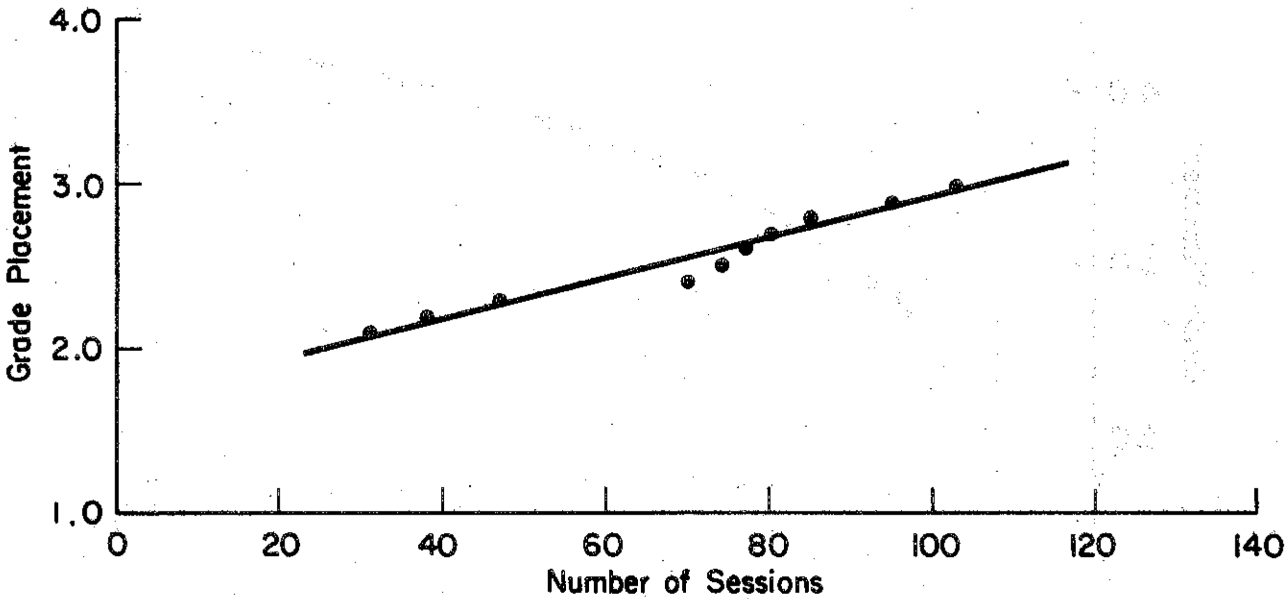

Fig. 2. Student with k = 1.00, b = .0122, and c = 1.69.

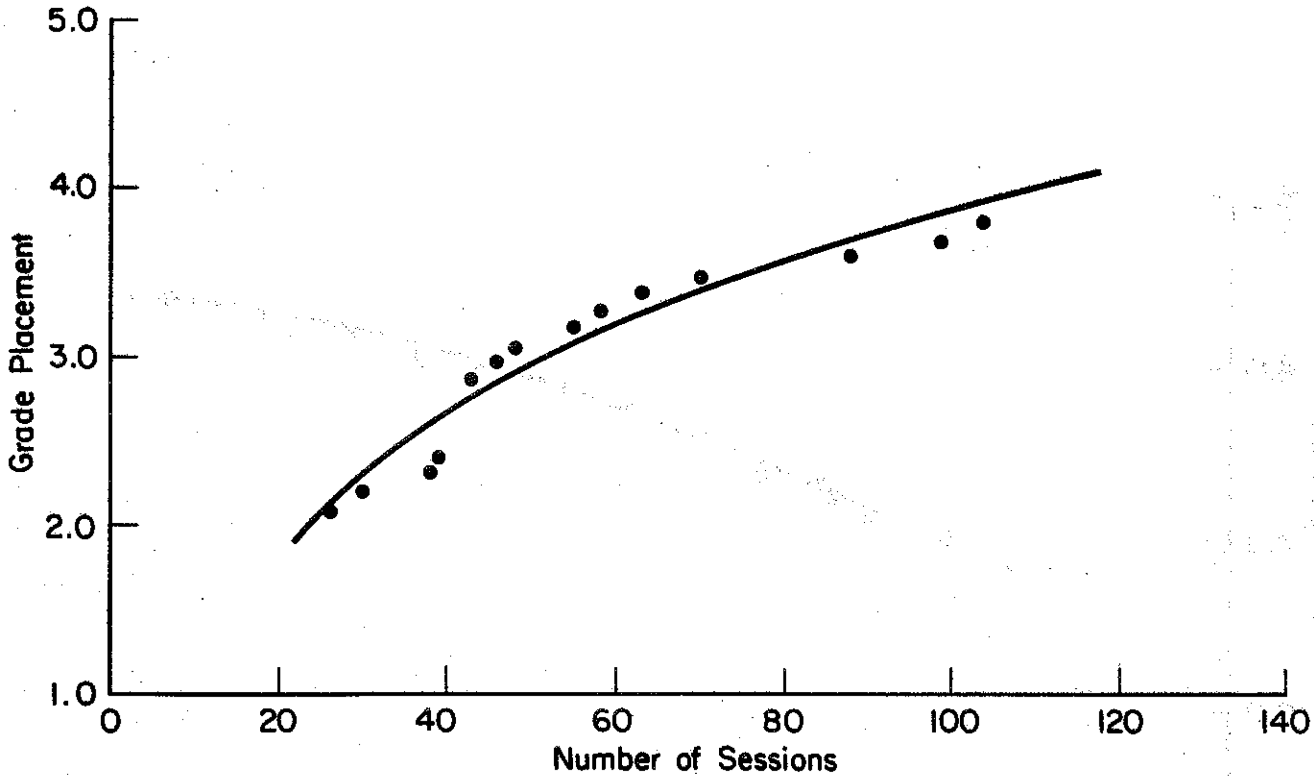

Fig. 3. Student with largest standard error.

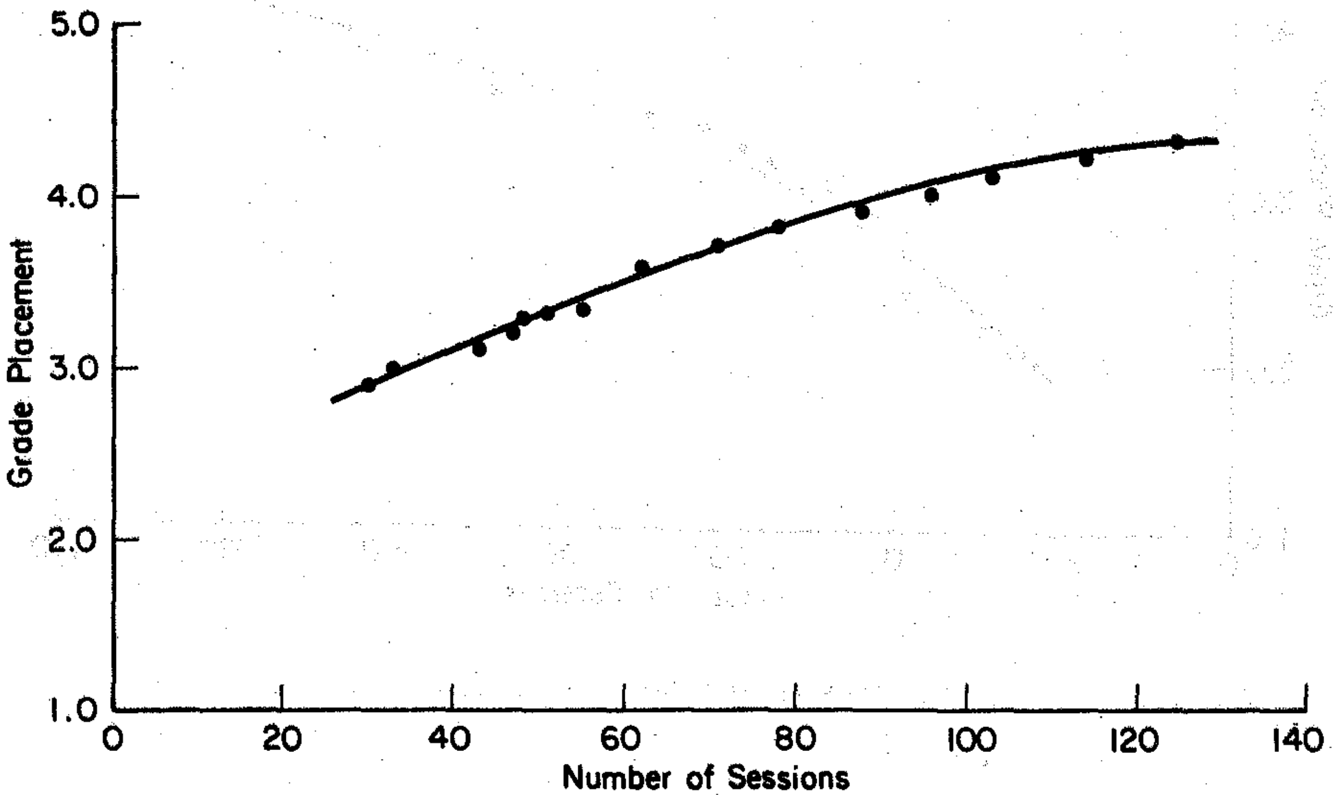

Fig. 4. Typical student with k = .40, b = .50, and c = .95.

It is apparent from the extremely close fits of the predicted curves to the data that these curves should be usable for predictive purposes.

The most important of the three estimated parameters for each student is the exponent k that enters in the basic equation. To give a sense of the effect of using the same k for all students and to see how the mean standard error varies with the variation of k, we show in Table 4 the results of letting k range from .05 to 1.00. We have also included the limiting case of the log model.

Insert Table 4 about here

As can be seen from this table, the mean standard error varies from .0856 for $k = 1.00$ to a minimum of .0602 for $k = .45$. In the third column we show the range across students of the standard error. Even in the worst case, that with $k = 1.00$, the top of the range is still only slightly more than one-quarter of a GP. The mean absolute residuals shown in the fourth column also have a relatively small value, running from a minimum of .0452 for $k = .40$ or $k = .45$ to a maximum of .0645 at $k = 1.00$. The ranges of the absolute maximum residuals, shown in the fifth column of Table 4, correspond closely to the ranges of the standard error.

At the bottom of the table we have shown the fixed value of $k = .47357$ that arises from taking the mean of the best individual k's. This mean fixed k is close to the minimum shown in the table in terms of standard error, that is, with a standard error of .0604; the same is true of the range of the standard error and the other data.

TABLE 4

Evaluation of Fit of Theory Using Same Exponent k for Each Student, but Individual Parameters b and c

k	Mean stand. error	Range SE	Mean abs. residual	Range abs. mean residual
ln	.0776	.0163-.2082	.0594	.0118-.1603
.05	.0740	.0148-.1994	.0565	.0104-.1620
.10	.0708	.0134-.2020	.0540	.0094-.1662
.15	.0679	.0124-.2047	.0516	.0086-.1704
.20	.0654	.0118-.2075	.0496	.0079-.1745
.25	.0634	.0117-.2105	.0480	.0079-.1787
.30	.0618	.0121-.2136	.0466	.0078-.1828
.35	.0608	.0128-.2168	.0457	.0083-.1869
.40	.0603	.0141-.2201	.0452	.0094-.1909
.45	.0602	.0158-.2235	.0452	.0107-.1949
.50	.0606	.0175-.2270	.0455	.0121-.1989
.55	.0616	.0161-.2305	.0462	.0117-.2028
.60	.0629	.0146-.2341	.0472	.0110-.2067
.65	.0646	.0134-.2377	.0486	.0097-.2105
.70	.0668	.0128-.2414	.0502	.0094-.2142
.75	.0693	.0128-.2452	.0521	.0092-.2179
.80	.0721	.0133-.2489	.0542	.0096-.2215
.85	.0752	.0143-.2527	.0566	.0107-.2251
.90	.0785	.0149-.2565	.0591	.0105-.2286
.95	.0819	.0142-.2603	.0618	.0097-.2320
1.00	.0856	.0136-.2641	.0645	.0090-.2354
.47357	.0604	.0166-.2251	.0452	.0114-.1968

In Table 5 we compare the results for this population mean of individually best k's with the mean standard error for the individually

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Insert Table 5 about here

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best k's, and we can see the improvement we get from going from approximately the best k that must be constant across students with individually estimated k's. The improvement in the mean standard error is significant, moving from .0604 down to .0458. There also is a corresponding improvement in the range as shown in the third column of Table 5, as well as a good improvement in the mean of the absolute residuals, moving from .0452 to .0343. A similar improvement obtains for the mean of the maximum absolute residuals.

Figure 5 shows how relatively flat the mean standard error is when a fixed parameter k is used for the entire student population; the data are graphed from the second column of Table 4. This figure shows

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Insert Figure 5 about here

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well enough that if a fixed k is used for the entire population there is no necessity to have a highly exact estimate of it. Any value in the range from .3 to .6 will give about as good an estimate as any other, with a possible improvement of not much more than two parts in a thousand.

When several parameters are estimated for each student, it is natural to ask what can be said about the joint distribution of the parameters. In the present case, perhaps the most interesting comparison is to use the mean fixed $k = .47357$ for the entire population and to study the properties of the joint distribution of the coefficients b and c.

TABLE 5

Comparison for Individually Best k's with Population Mean of Individually Best k's

	Mean stand. error	SD of SE	Range of SE	Mean of mean abs. residuals	Mean of max. abs. residuals
Mean k = .47357	.0604	.0280	.0166-.2251	.0452	.1042
Individual k's	.0458	.0208	.0117-.1970	.0343	.0803

Note.--In both cases, parameters b and c are individual parameters.

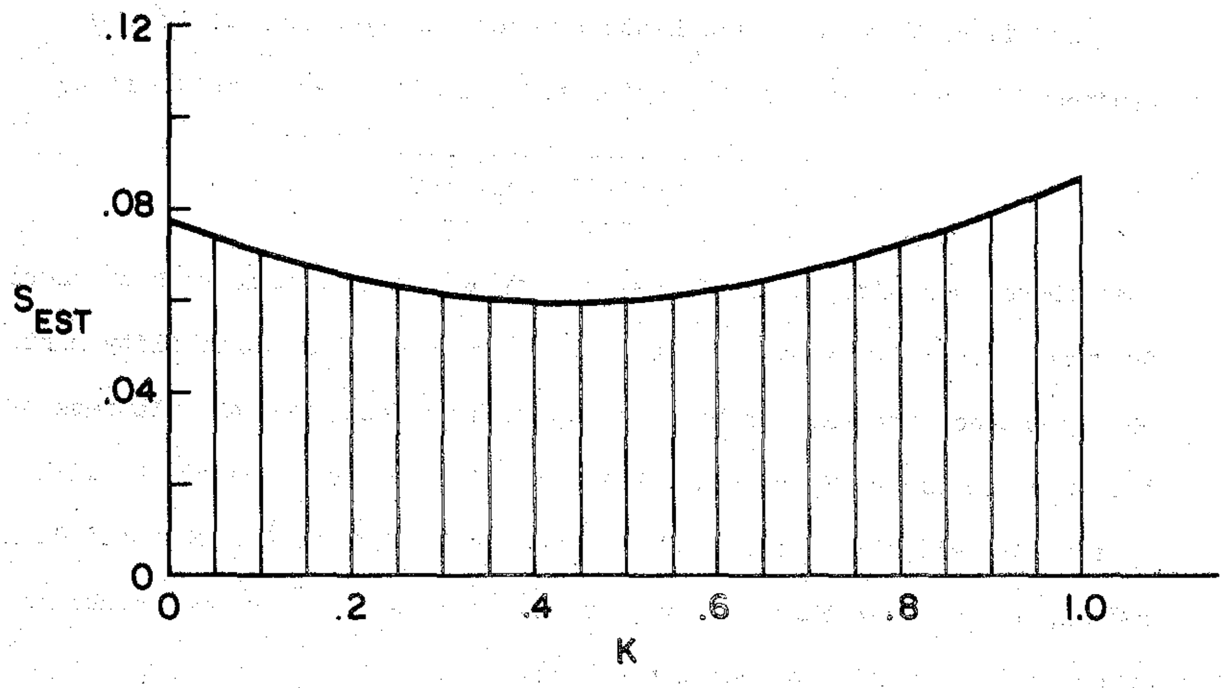

Fig. 5. Graph of mean standard error as a function of the parameter k.

The scatter plot of this joint distribution is shown in Figure 6. As is evident from the figure, there is a negative correlation between the two

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Insert Figure 6 about here

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coefficients, with $r = -.5772$. The absolute value of the correlation is low enough to show that we cannot eliminate one of the coefficients and achieve as good predictive results.

In Figure 7, we show the histogram for the distribution of the exponent k when individually estimated. It is clear from the figure

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Insert Figure 7 about here

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that there is a wide range of best k values, and in this respect there is great student variability. On the other hand, this variability must be approached with caution because, as we have seen from the flatness of the curve in Figure 5, considerable variation in the range of k will affect only slightly the fit of the predicted curve to the observed data. Indeed, it is clear that even with fixed $k = .47357$ the mean standard error is well within an acceptable limit.

DISCUSSION

From some simple and unquestionably too schematic assumptions about information processing, we have derived a stochastic differential equation for the motion of a student through a CAI elementary mathematics curriculum. The constants of integration were estimated for each student individually, and a reasonable fit of the theory to the data was obtained

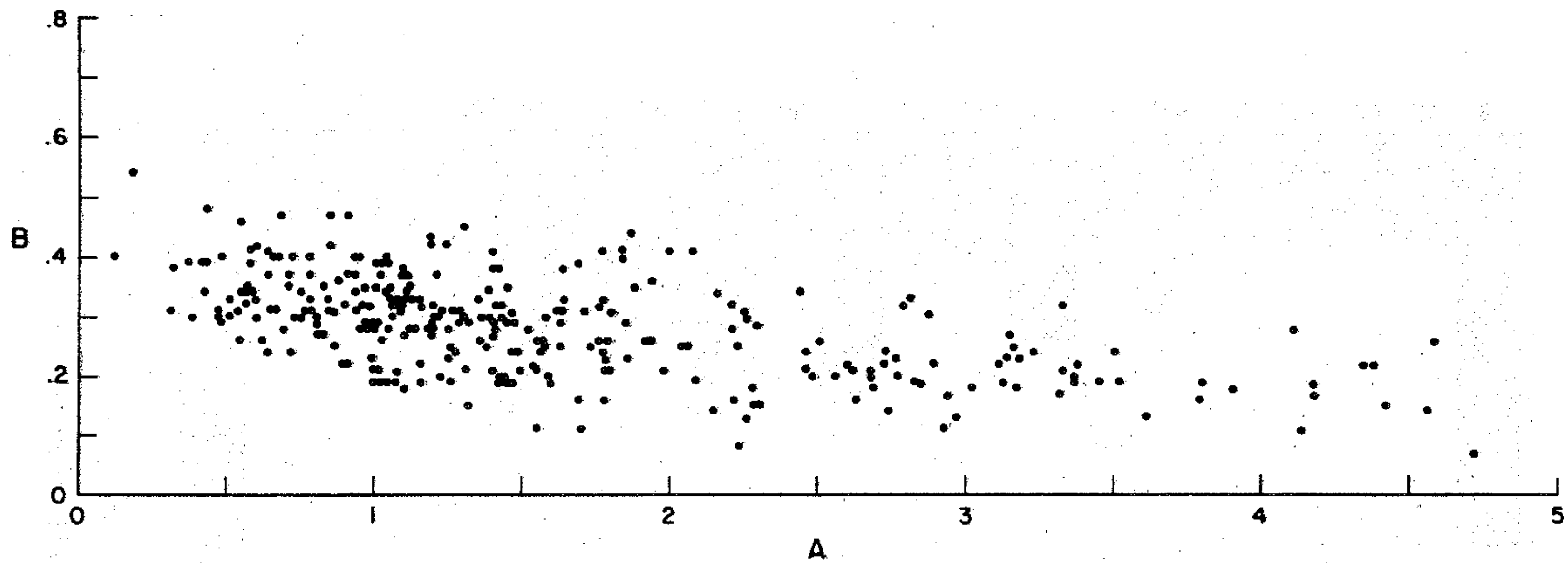

Fig. 6. Scatter plots of individual parameter pairs (b,c) with k = .47357.

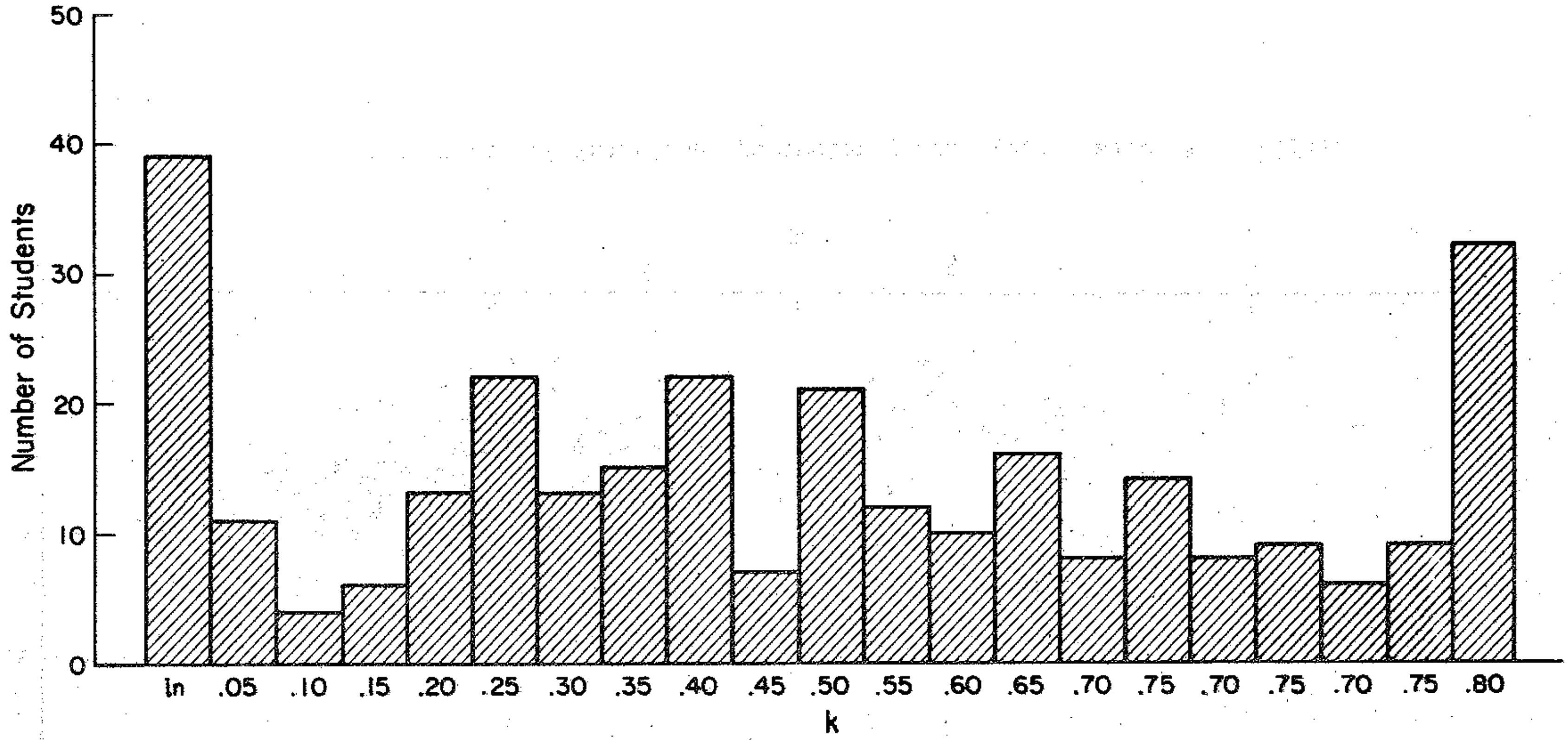

Fig. 7. Histogram of the exponent k individually estimated for 297 students.

in terms of mean standard error. We believe that the kind of global model exemplified in this work has an important, but, as yet, generally unrecognized contribution to make to educational psychology. Most of the quantitative research in educational psychology has been concerned with the microscopic processing of items by students, or with the characteristics of tests. Without doubt, much has been accomplished in both of these areas--the first in terms of learning theory and the second in terms of test theory. What has been missing is a dynamical theory of a student's broad progress through a given curriculum. What we have attempted to provide in the present report is a test of a dynamical equation of motion derived from qualitative principles. We hope the theory will be tested in other areas of the curriculum. We recognize that one of the difficulties of application is making the kind of detailed analysis of curriculum that lies back of the strands mathematics curriculum used in the present study.

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FOOTNOTES

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