NOTE ON COMPUTING ALL OPTIMAL SOLUTIONS OF A DUAL LINEAR PROGRAMMING PROBLEM

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In certain applications of linear programming it may be desired to find the set of all optimal solutions to the dual problem.[1/] The instructions given in Charnes, Cooper and Henderson [1] are at best vague. Using the notation of [1] the purpose of this note is to state an explicit algorithm. We begin with two simple lemmas.

Lemma 1. *In every tableau for* $\ell = 1,\ldots,m$ *and* $j = 1,\ldots,n$

$$x_{\ell j} = \sum_{i=1}^{m} x_{\ell,n+i} A_{ij} .$$

Proof: Trivial for the initial tableau. Assume then that it holds for the n^{th} tableau. For $\ell \neq k$ and $j = 1,\ldots,m+n$

$$x'_{\ell j} = x_{\ell j} - \frac{x_{rj}}{x_{rk}} x_{\ell k} .$$

Hence to show that

$$x'_{\ell j} = \sum_{i=1}^{m} x'_{\ell,n+i} A_{ij} \qquad j = 1,\ldots,n$$

we need to show that

1/ My own interest stems from applying linear programming methods to problems of psychological scaling. This research was supported by the Office of Naval Research under Contract NR 171-034, Group Psychology Branch.

$$\frac{x_{rj}}{x_{rk}} x_{\ell k} = \sum_{i=1}^{m} \frac{x_{r,n+i}}{x_{rk}} x_{\ell k} A_{ij} ,$$

that is, that

$$x_{rj} = \sum_{i=1}^{m} x_{r,n+i} A_{ij} ,$$

but this follows from our inductive hypothesis.

For $\ell = k$

$$x'_{kj} = \frac{x_{rj}}{x_{rk}} \qquad j = 1,\ldots,m+n ,$$

and the argument is exactly similar.

<u>Lemma</u> 2. <u>In every tableau for</u> $j = 1,\ldots,n$

$$z_j = \sum_{i=1}^{m} z_{n+i} A_{ij} .$$

<u>Proof:</u> Trivial for initial tableau. Assume that it holds for n^{th} tableau. Now

$$z'_j = z_j - \frac{x_{rj}}{x_{rk}} (z_k - c_k) .$$

Hence by virtue of our inductive hypothesis, after making the obvious cancellation of terms, we need only to show that

$$x_{rj} = \sum_{i=1}^{m} x_{r,n+i} A_{ij} ,$$

but this follows at once from Lemma 1.

We next explicitly define distinct optimal solutions of the dual problem in terms of the simplex tableaus.

Definition. Let T and T' be two optimal tableaus. Then T and T' yield distinct optimal solutions of the dual problem if, and only if, there is a j with $n < j \leq n+m$ such that

$$z_j \neq z_j' .$$

We now state the theorem which provides an algorithm for finding such distinct solutions or deciding there are none. I say that tableau T' is immediately derived from tableau T when T' is computed directly from T according to the procedure given in [1].

Theorem. Let T be an optimal tableau. In order that there exist an optimal tableau T' immediately derivable from T and yielding a distinct optimal solution to the dual problem it is necessary and sufficient that there is a row r and a column k in T such that

(1) $\lambda_r = 0$

(2) $x_{rk} \neq 0$

(3) $z_k - c_k > 0$

(4) For every column j in T

$$z_j - c_j \geq \frac{x_{rj}}{x_{rk}} (z_k - c_k) .$$

Proof: [Necessity]. It is convenient to consider the conditions in reverse order. If there is a j such that

$$\frac{x_{rj}}{x_{rk}}(z_k - c_k) > z_j - c_j ,$$

then

$$z_j' - c_j' < 0 ,$$

and T' is not an optimal tableau.

By hypothesis on T, $z_k - c_k \geq 0$. If $z_k - c_k = 0$ then for all columns j in T

$$z_j' - c_j' = z_j - c_j$$

and T' does not yield a distinct solution.

If $x_{rk} = 0$ then T' is not well-defined, since the computation for T' will involve division by zero.

If $\lambda_r \neq 0$, since $x_{rk} \neq 0$ and $z_k - c_k > 0$, we must have

$$z_o \neq z_o' ,$$

but then T and T' cannot both be optimal, for z_o is the optimal value of the linear functional.

[Sufficiency]. Suppose the conditions (1)-(4) are satisfied and there is no i with $1 \leq i \leq m$ such that

$$z_{n+i} \neq z_{n+i}' .$$

Now from our hypothesis we know there is a k such that

$$z_k' \neq z_k$$

for

$$z_k - c_k > 0$$

and always

$$z'_k - c'_k = 0 .$$

On our supposition k is not the number of a slack vector, that is, $1 \leq k \leq n$. Hence by virtue of Lemma 2 we know that

$$\sum_{i=1}^{m} z_{n+i} A_{ik} \neq \sum_{i=1}^{m} z'_{n+i} A_{ik} ,$$

but this inequality can only hold if for some i with $1 \leq i \leq m$

$$z_{n+i} \neq z_{n+i} ,$$

which is contrary to our supposition. This completes the proof of the theorem.

Given the theorem, <u>all</u> optimal solutions to the dual problem may be found by the Tarry method outlined on pp. 69-70 of [1]. It is perhaps useful to note that when $z_k - c_k = 0$ we may bring the vector k into the basis to find a new solution to the original problem but as condition (3) of the theorem shows, this tableau <u>cannot</u> yield a new solution to the dual problem. The remark on p. 21 of [1] that to obtain all optimal solutions we need only consider the case of $z_k - c_k = 0$ and the case of $\lambda_i = 0$ associated with $x_{ij} > 0$ is misleading, for as the theorem given here indicates we may have $x_{ij} < 0$ and obtain a new solution to the dual problem. My own experience has been that this is more common than having $x_{ij} > 0$.

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References

1. A Charnes, W.W. Cooper and A. Henderson, An Introduction to Linear Programming, New York, 1953.

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