NOTE ON COMPUTING ALL OPTIMAL SOLUTIONS OF A
DUAL LINEAR PROGRAMMING PROBLEM

BY

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In certain applications of linear programming it may be desired to find the set of all optimal solutions to the dual problem. The instructions given in Charnes, Cooper and Henderson [1] are at best vague. Using the notation of [1] the purpose of this note is to state an explicit algorithm. We begin with two simple lemmas.

Lemma 1. In every tableau for \( l = 1, \ldots, m \) and \( j = 1, \ldots, n \)

\[
x'_{l,j} = \sum_{i=1}^{m} x_{l,n+1} A_{i,j}
\]

Proof: Trivial for the initial tableau. Assume then that it holds for the \( n \)th tableau. For \( l \neq k \) and \( j = 1, \ldots, m+n \)

\[
x'_{l,j} = x'_{l,j} - \frac{x_{l,j}}{x_{l,k}} x'_{l,k}
\]

Hence to show that

\[
x'_{l,j} = \sum_{i=1}^{m} x'_{l,n+1} A_{i,j} \quad j = 1, \ldots, n
\]

we need to show that

\[1/\] My own interest stems from applying linear programming methods to problems of psychological scaling. This research was supported by the Office of Naval Research under Contract NR 171-034, Group Psychology Branch.
\[
\frac{x_{rj}}{x_{rk}} x_{j,k} = \sum_{i=1}^{m} \frac{x_{r,n+i}}{x_{rk}} x_{j,k} A_{ij},
\]

that is, that

\[
x_{rj} = \sum_{i=1}^{m} x_{r,n+i} A_{ij},
\]

but this follows from our inductive hypothesis.

For \( j = k \)

\[
x_{k,j} = \frac{x_{rj}}{x_{rk}} \quad j = 1, \ldots, m+n,
\]

and the argument is exactly similar.

**Lemma 2.** In every tableau for \( j = 1, \ldots, n \)

\[
z_j = \sum_{i=1}^{m} z_{n+i} A_{ij}.
\]

**Proof:** Trivial for initial tableau. Assume that it holds for \( n \)th tableau. Now

\[
z_j = z_j - \frac{x_{rj}}{x_{rk}} (z_k - c_k).
\]

Hence by virtue of our inductive hypothesis, after making the obvious cancellation of terms, we need only to show that

\[
x_{rj} = \sum_{i=1}^{m} x_{r,n+i} A_{ij},
\]

but this follows at once from Lemma 1.
We next explicitly define distinct optimal solutions of the dual problem in terms of the simplex tableaus.

**Definition.** Let $T$ and $T'$ be two optimal tableaus. Then $T$ and $T'$ yield distinct optimal solutions of the dual problem if, and only if, there is a $j$ with $n < j < n+m$ such that

$$z_j 
eq z'_j.$$

We now state the theorem which provides an algorithm for finding such distinct solutions or deciding there are none. I say that tableau $T'$ is immediately derived from tableau $T$ when $T'$ is computed directly from $T$ according to the procedure given in [1].

**Theorem.** Let $T$ be an optimal tableau. In order that there exist an optimal tableau $T'$ immediately derivable from $T$ and yielding a distinct optimal solution to the dual problem it is necessary and sufficient that there is a row $r$ and a column $k$ in $T$ such that

1. $\lambda_k = 0$
2. $x_{rk} 
eq 0$
3. $z_k - c_k > 0$
4. For every column $j$ in $T$

$$z_j - c_j \geq \frac{x_{rj}}{x_{rk}} (z_k - c_k).$$

**Proof: ** (Necessity). It is convenient to consider the conditions in reverse order. If there is a $j$ such that
then
\[
\frac{x_j}{x_{rk}} (z_k - c_k) > z_j - c_j,
\]
then
\[
z'_j - c'_j < 0,
\]
and \( T' \) is not an optimal tableau.

By hypothesis on \( T \), \( z_k - c_k > 0 \). If \( z_k - c_k = 0 \) then for all columns \( j \) in \( T \)
\[
z'_j - c'_j = z_j - c_j
\]
and \( T' \) does not yield a distinct solution.

If \( x_{rk} = 0 \) then \( T' \) is not well-defined, since the computation for \( T' \)
will involve division by zero.

If \( x_{rk} \neq 0 \), since \( x_{rk} \neq 0 \) and \( z_k - c_k > 0 \), we must have
\[
z'_0 \neq z_0
\]
but then \( T \) and \( T' \) cannot both be optimal, for \( z_0 \) is the optimal value of
the linear functional.

[Sufficiency]. Suppose the conditions (1)-(4) are satisfied and there is
no \( i \) with \( 1 \leq i \leq m \) such that
\[
z_{n+i} \neq z'_{n+i}.
\]
Now from our hypothesis we know there is a \( k \) such that
\[
z'_k \neq z_k
\]
for
\[
z_k - c_k > 0
\]
and always

\[ z'_{k} - c'_{k} = 0. \]

On our supposition \( k \) is not the number of a slack vector, that is, \( 1 \leq k \leq n \). Hence by virtue of Lemma 2 we know that

\[ \sum_{i=1}^{m} z_{n+i} A_{ik} / \neq \sum_{i=1}^{m} z'_{n+i} A'_{ik}, \]

but this inequality can only hold if for some \( i \) with \( 1 \leq i \leq m \)

\[ z_{n+i} / \neq z'_{n+i}, \]

which is contrary to our supposition. This completes the proof of the theorem.

Given the theorem, all optimal solutions to the dual problem may be found by the Tarry method outlined on pp. 69-70 of [1]. It is perhaps useful to note that when \( z_{k} - c_{k} = 0 \) we may bring the vector \( k \) into the basis to find a new solution to the original problem but as condition (3) of the theorem shows, this tableau cannot yield a new solution to the dual problem. The remark on p. 21 of [1] that to obtain all optimal solutions we need only consider the case of \( z_{k} - c_{k} = 0 \) and the case of \( \lambda_{i} = 0 \) associated with \( x_{i,j} > 0 \) is misleading, for as the theorem given here indicates we may have \( x_{i,j} < 0 \) and obtain a new solution to the dual problem. My own experience has been that this is more common than having \( x_{i,j} > 0 \).

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