

A LINEAR LEARNING MODEL FOR A CONTINUUM OF RESPONSES

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## 1. Introduction.

The aim of the present investigation is to formulate and analyze a linear model for simple learning with a continuum of responses. The analogous model for a finite number of responses has been extensively studied both experimentally and mathematically (for references, see Estes and Suppes, Technical Report No. 16 of this series).

The experimental situation consists of a sequence of trials. On each trial the subject (of the experiment) makes a response, which is followed by a reinforcing event indicating the correct response for that trial. For the theory we are considering the relevant data on each trial may be represented by an ordered pair  $(x,y)$  of numbers. The number  $x$  represents the response and  $y$  the reinforcement. In the finite case of two responses and two reinforcements, we have simply that  $x$  and  $y$  are either 0 or 1. (The phrases "finite case" and "finite model" refer to the linear model for a finite number of responses; and the phrases "continuous case" and "continuous model" to the linear model for a continuum of responses. The psychological basis of the finite and continuous models is the same, namely, the fundamental postulate that in a simple learning situation with the same stimulus presentation on each

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trial, systematic changes in response behavior are determined by the experimental conditions of reinforcement.

As far as I know there have been few if any learning experiments of a quantitative character involving a continuum of both responses and reinforcements on each trial. However, once the question of performing such experiments is raised, it is very easy to think of a variety of specific experimental situations to test the model to be described here. In order to give the developments to follow a certain concreteness, I will describe at this point and later consider in some numerical detail the apparatus now being used at Stanford University to study the empirical validity of this continuous model. The subject is seated facing a circular disc, which is approximately 20 inches in diameter. He is told that his task on each trial is to predict by means of a pointer where a spot of light will appear on the rim of the disc. At the end of each trial the "actual" position of the spot is shown to the subject, which is the reinforcing event for that trial. The objective of the model considered here is the same as that of the finite models already much tested, namely, to predict for each trial, and asymptotically, the probability distribution of responses. (The subject's responses are his pointer predictions.) In the case of the disc, the response number  $x$  and the reinforcement number  $y$  vary continuously from 0 to  $2\pi$ .

From the character of this example of a typical experiment to which the model applies, it should be obvious that the model is also relevant to a large number of skill-performance tasks where the independent and dependent variables

are both continuous. On the other hand, as the model is formulated in this paper it is not meant to apply directly to measures of response amplitude, where responses and reinforcements are themselves discrete.

For either the finite or continuous case an experiment may be represented by a sequence  $(A_1, E_1, A_2, E_2, \dots, A_n, E_n \dots)$  of random variables, where the choice of letters follows conventions established in the literature: the value of the random variable  $A_n$  is the number representing the actual response on trial  $n$ , and the value of  $E_n$  is the number representing the reinforcing event which occurred on trial  $n$ . Any sequence of values of these random variables represents a possible experimental outcome. Hereafter, we write down only finite sequences of these random variables and it is convenient to write them in reverse order:  $(E_n, A_n, E_{n-1}, A_{n-1}, \dots, E_1, A_1)$ .

For both the finite and continuous model the theory is formulated for the probability of a response on trial  $n+1$  given the entire preceding sequence of responses and reinforcements. For this preceding sequence we use the notation  $s_n$  <sup>\*</sup>. Thus, more formally,  $s_n$  is a finite sequence of length  $2n$  of possible values of the sequence of random variables  $(E_n, A_n, E_{n-1}, A_{n-1}, \dots, E_1, A_1)$ . The (cumulative) joint distribution of the first  $n$  responses and  $n$  reinforcements is denoted by  $J_n$ , that is, if  $s_n = (y_n, x_n, \dots, y_1, x_1)$ , then

$$(1.1) \quad J_n(s_n) = J_n(y_n, x_n, \dots, y_1, x_1) = P(E_n \leq y_n, A_n \leq x_n, \dots, E_1 \leq y_1, A_1 \leq x_1)$$

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<sup>\*</sup>/ The notation  $s_n$  corresponds to  $x_n$  in Estes and Suppes, Technical Report No. 16 of this series. Hereafter we refer to this report as ES.

(where  $P$  is the measure on the underlying sample space). For notational clarity we use variables  $x_1, x_2, \dots, x_n, \dots$  for values of the response random variables and  $y_1, y_2, \dots, y_n, \dots$  for the reinforcement random variables.

Restricting ourselves in the finite case to the values 0 and 1 for the random variables, we define

$$(1.2) \quad p_{1,n} = P(A_n = 1) ,$$

$$(1.3) \quad p_{s,1,n} = P(A_n = 1 \mid s_{n-1}) .$$

The linear model for the two-response case is embodied in the following two axioms (a more general formulation requiring more axioms is possible (cf. ES, Section 3) but will not be considered here).

Axiom F1. If  $E_n = 1$  , then

$$p_{s,1,n+1} = (1 - \theta)p_{s,1,n} + \theta .$$

Axiom F2. If  $E_n = 0$  , then

$$p_{s,1,n+1} = (1 - \theta)p_{s,1,n} .$$

Here  $\theta$  is intuitively to be thought of as the learning parameter; formally  $\theta$  is some real number in the open interval  $(0,1)$  .

There are exactly three results for this finite model that we want briefly to mention. The first is that we may derive for the "mean" probabilities  $p_{1,n}$  the following recurrence relation:

$$(1.4) \quad p_{1,n+1} = (1 - \theta)p_{1,n} + \theta P(E_n = 1) .$$

A variety of testable experimental results follow from specifying  $P(E_n = 1)$ . The case most studied is the so-called non-contingent one where we set  $P(E_n = 1) = \pi$ , where  $\pi \in (0,1)$  and is independent of  $n$ . For this case (1.4) has the explicit solution:

$$(1.5) \quad p_{1,n} = \pi - (\pi - p_{1,1})(1 - \theta)^{n-1}$$

and the asymptotic result:

$$(1.6) \quad p_{1,\infty} = \pi,$$

that is, the asymptotic probability of a response is equal to the non-contingent probability of a reinforcement.

The third result deals with the simple contingent case, specified by the following conditional probabilities of reinforcement:

$$(1.7) \quad \begin{cases} P(E_n = 1 \mid A_n = 1) = \pi_1 \\ P(E_n = 1 \mid A_n = 0) = \pi_0 \end{cases}.$$

Again the asymptotic result for this special case of (1.4) is easily established:

$$(1.8) \quad p_{1,\infty} = \frac{\pi_0}{1 - \pi_1 + \pi_0}.$$

A number of further statistically testable dependency relations and sequential effects for both the non-contingent and the contingent case may be derived from Axioms F1 and F2 (cf. Section 8 of ES), but the analogues of (1.4), (1.5), (1.6) and (1.8) will be initially sufficient for discussing the continuous case.

We need first some further notation for various distributions. It is assumed that the values of all the random variables lie in some interval  $[a,b]$  (with possibly  $a = -\infty$  and  $b = \infty$ ). In addition to the joint distribution  $J_n$ , we denote the marginal distribution of  $A_n$  by:

$$(1.9) \quad R_n(x) = P(A_n \leq x) ,$$

and the marginal distribution of  $E_n$  by

$$(1.10) \quad F_n(y) = P(E_n \leq y) .$$

We shall also use  $F$  for certain conditional distributions of  $E_n$ . Densities (which as a consequence of our axioms will always exist) are denoted by the corresponding lower-case letters.

## 2. Axioms.

In the continuous case we also have a learning parameter  $\theta$ , but it does not seem reasonable for the full effect of reinforcement to be concentrated at a point as Axiom F1 asserts it is for the finite case. Consequently we assume in addition to  $\theta$ , a smearing distribution  $K(x;y)$  which spreads the effect of reinforcement around the point of reinforcement. For each reinforcement  $y$ ,  $K(x;y)$  is a distribution on responses, that is,

$$K(a;y) = 0$$

$$K(b;y) = 1$$

and if  $x \leq x'$  , then

$$K(x;y) \leq K(x';y) ,$$

for every  $y$  in  $[a,b]$  . The reasonable assumptions that the density  $K(x;y)$  is unimodal (at  $y$ ) and that it is symmetric will be investigated in the sequel but not assumed as fundamental axioms. Prior at least to extensive experimentation these assumptions seem most plausible empirically.

The first two axioms are simply directed at making explicit assumptions of smoothing properties which seem highly justified empirically. Mathematically weaker assumptions could be made but this is not a matter of importance here. The third axiom asserts the analogue of Axioms F1 and F2 for the finite case.

Axiom C1. The distribution  $J_n$  is continuous and piecewise twice differentiable in each variable.

Axiom C2. The distribution  $K(x;y)$  is continuous and piecewise twice differentiable in both variables.

Axiom C3.

$$J_{n+1}(x | y_n, x_n, s_{n-1}) = (1 - \theta)J_n(x | s_{n-1}) + \theta K(x; y_n) .$$

Axiom C3 says that given the sequence  $(y_n, x_n, s_{n-1})$  then the conditional distribution of the response random variable on trial  $n+1$  is  $(1 - \theta)$  times the conditional distribution of the response random variable on trial  $n$  given the sequence  $s_{n-1}$  plus  $\theta$  times the smearing distribution  $K(x; y_n)$  .



The parameter of the smearing distribution is, of course, the point of reinforcement  $y_n$  in the sequence  $(y_n, x_n, s_{n-1})$ .

3. Derivation of Integral Equations for Asymptotic Response Distributions.

In this section we want to derive continuous analogues of (1.4), (1.5), (1.6) and (1.8).

As the analogue of (1.4), we have

Theorem

$$(3.1) \quad R_{n+1}(x) = (1 - \theta)R_n(x) + \theta \int_a^x \int_a^b k(x;y)f_n(y)dx dy$$

and if  $a \leq a_1 < a_2 \leq b$ , then

$$(3.2) \quad \int_{a_1}^{a_2} r_{n+1}(x)dx = (1 - \theta) \int_{a_1}^{a_2} r_n(x)dx + \theta \int_{a_1}^{a_2} \int_a^b k(x;y)f_n(y)dx dy .$$

Proof:

$$\begin{aligned} R_{n+1}(x) &= \int_a^x r_{n+1}(x)dx \\ &= \int_a^x \underbrace{\int_a^b \dots \int_a^b}_{2n \text{ times}} j_{n+1}(x, s_n) dx ds_n \end{aligned}$$

by the definition of the marginal distribution  $R_n$

$$= \int_a^x \int_a^b \dots \int_a^b j_{n+1}(x | y_n, x_n, s_{n-1}) j_n(y_n, x_n, s_{n-1}) dx dy_n dx_n ds_{n-1}$$

by usual conditional distribution relations

$$= \int_a^x \int_a^b \dots \int_a^b [(1-\theta)j_n(x | s_{n-1}) + \theta k(x; y_n)] \cdot j_n(y_n, x_n, s_{n-1}) dx dy_n dx_n ds_{n-1}$$

by applying Axiom C3 in terms of densities

$$= (1-\theta)R_n(x) + \theta \int_a^x \int_a^b k(x; y_n) f_n(y_n) dy_n$$

by integrating out.

And (3.2) easily follows from (3.1). Q.E.D.

From (3.1) we easily obtain the non-contingent analogues of (1.5) and (1.6).

Here  $R(x)$  is the mean asymptotic probability distribution of responses, and

$F(y)$  is the reinforcement distribution which is independent of  $n$ .

Theorem. In the non-contingent case

$$(3.3) \quad R_n(x) = \int_a^x \int_a^b k(x; y) f(y) dx dy - \left[ \int_a^x \int_a^b k(x; y) f(y) dx dy - R_1(x) \right] (1-\theta)^{n-1}$$

and

$$(3.4) \quad \lim_{n \rightarrow \infty} R_n(x) = R(x) = \int_a^x \int_a^b k(x; y) f(y) dx dy$$

Proof: Since the non-contingent case is specified by a distribution  $F(y)$  independent of  $n$ , (3.1) becomes

$$R_{n+1}(x) = (1-\theta)R_n(x) + \theta \int_a^x \int_a^b k(x; y) f(y) dx dy$$

The integral on the right is a function  $c(x)$  of  $x$  alone. Thus the equation may be written:

$$R_{n+1}(x) = (1 - \theta)R_n(x) + c(x) ,$$

and exactly the argument which yields (1.5) and (1.6) in the finite case yields (3.3) and (3.4) as consequences of this last equation. Q.E.D.

The integral equation (3.4) for the asymptotic response distribution calls for some comment. If the smearing distribution  $K(x;y)$  is known, then the response distribution  $R(x)$  may be obtained by direct integration, since the reinforcement distribution  $F(y)$  is selected by the experimenter and is thus completely known. A central difficulty, however, is that the smearing distribution is not directly observable. By this I mean there is no observable quantity in the experimental set-up which yields an empirical histogram of this distribution. It is possible to infer the smearing distribution from  $R(x)$  and  $F(y)$  by using (3.4). From this standpoint, if we assume that the density  $k(x;y)$  is a symmetric function of the difference  $x - y$ , we may differentiate (3.4), make a change of variable and get:

$$(3.5) \quad r(x) = \int_{x-a}^{x-b} k(t)f(x-t)dt$$

which is an integral equation of Fredholm type with kernel  $f(x-t)$ . Techniques for solving (3.5) by using the empirical histogram of  $r(x)$  will not be pursued here, but it may be noted that by approximating  $r(x)$  and  $f(x-t)$  by means of trigonometric polynomials a good approximation of  $k(t)$  may be obtained under relatively mild restrictions on degeneracy.

Because the smearing distribution is postulated to be independent of the reinforcement distribution, the solution of  $k(t)$  from (3.5) for one experiment may be used to predict the response distribution in other experiments with different reinforcement distributions.

We now turn to analogues of the simple contingent case specified by (1.7) for the finite model. For the simplest analogue we choose a point  $c$  in  $(a,b)$  and if the response  $x \leq c$  is made, we use distribution  $f_1$  for reinforcements and if response  $x > c$  is made, we use distribution  $f_2$ . That is, the simple contingent case is specified by a triple  $(c, f_1, f_2)$  such that

$$(3.6) \quad f_n(y) = \int_a^c f_1(y)r_n(x)dx + \int_c^b f_2(y)r_n(x)dx .$$

Since  $R_n(a) = 0$  and  $R_n(b) = 1$ , we may integrate (3.6) to obtain

$$(3.7) \quad f_n(y) = R_n(c)f_1(y) + (1 - R_n(c))f_2(y) .$$

We may use (3.7) to obtain the asymptotic response distribution:

Theorem. For the simple contingent case as defined by (3.6)

$$(3.8) \quad R(x) = \int_a^x \int_a^b k(x;y)[R(c)f_1(y) + (1 - R(c))f_2(y)]dx dy ,$$

where

$$(3.9) \quad R(c) = \frac{\int_a^c \int_a^b k(x;y)f_2(y)dx dy}{1 - \int_a^c \int_a^b k(x;y)f_1(y)dx dy + \int_a^c \int_a^b k(x;y)f_2(y)dx dy}$$

Proof: Putting (3.7) into (3.1) we get:

$$(3.10) \quad R_{n+1}(x) = (1-\theta)R_n(x) + \theta \int_a^x \int_a^b k(x;y)[R_n(c)f_1(y) + (1-R_n(c))f_2(y)] dx dy .$$

By rearranging the second term on the right of (3.10), and letting  $x = c$ , we have:

$$(3.11) \quad R_{n+1}(c) = (1-\theta)R_n(c) + \theta[R_n(c)g_1(c) + g_2(c)] ,$$

where

$$g_1(c) = \int_a^c \int_a^b k(x;y)[f_1(y) - f_2(y)] dx dy$$

and

$$g_2(c) = \int_a^c \int_a^b k(x;y)f_2(y) dx dy .$$

Now (3.11) may be rewritten:

$$R_{n+1}(c) = (1-\theta + \theta g_1(c))R_n(c) + \theta g_2(c) ,$$

and it follows easily from this difference equation that

$$R(c) = \lim_{n \rightarrow \infty} R_n(c) = \frac{g_2(c)}{1-g_1(c)} ,$$

which is precisely (3.9), as was desired. Moreover, given (3.9), (3.8)

follows by a simple argument from (3.10). Q.E.D.

It is to be noted that (3.9) has exactly the same form as (1.8). More importantly, (3.8) has the same form as (3.4) for the non-contingent case, since

$$R(c)f_1(y) + (1 - R(c))f_2(y)$$

is a proper density function. Consequently the same mathematical techniques apply to solving the integral equation for the smearing distribution.

More complicated contingent cases are easily constructed. For example, a straightforward generalization of the results just obtained is the division of the integral  $[a,b]$  into an arbitrary finite number of subintervals rather than simply two. For this situation the experimenter selects  $r$  reinforcement distributions, one for each subinterval. Another example, further removed from the finite model, is the use by the experimenter of a continuous conditional reinforcement distribution  $f(y | x)$ . A conditional distribution of some interest would be one which, in the case of the circular disc mentioned earlier, would have a mean reinforcement so many degrees clockwise or counterclockwise from the response.

The continuous contingent case is formally defined by:

$$(3.12) \quad f_n(y) = \int_a^b f(y | x) r_n(x) dx .$$

It is possible to establish the existence of the asymptotic response distribution for the continuous contingent case by considering this case as the limit of a properly selected sequence of simple contingent cases with an increasing

number of subintervals. The argument in terms of upper and lower bounds is standard and yields the following theorem (we omit a detailed proof):

Theorem. For the continuous contingent case as defined by (3.12),

$$(3.13) \quad R(x) = \int_a^x \int_a^b \int_a^b k(x;y)f(y|t)r(t)dxdt dy .$$

#### 4. Some Numerical Examples.<sup>\*/</sup>

It should be useful to examine the form of the asymptotic response distributions when simple non-contingent reinforcement distributions are selected and it is assumed that the smearing distribution has some analytically elementary form. All of the examples to be considered refer to the circular disc apparatus mentioned in Section 1.

Example 1. Parabolic smearing distribution. We take as the density of the smearing distribution the parabola:

$$(4.1) \quad k(x-y) = \begin{cases} \left(\frac{3}{4}\right)^{2/3} - (x-y)^2 & \text{for } |x-y| \leq \left(\frac{3}{4}\right)^{1/3} \\ 0 & \text{for } \left(\frac{3}{4}\right)^{1/3} < |x-y| \leq \pi \end{cases}$$

For the reinforcement distribution, we use a simple linear density function whose max is at  $\pi$  :

$$(4.2) \quad f(y) = \begin{cases} \frac{y}{\pi} & \text{for } 0 \leq y \leq \pi \\ \frac{2}{\pi} - \frac{y}{\pi} & \text{for } \pi < y < 2\pi . \end{cases}$$

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<sup>\*/</sup> I am indebted to Gordon Latta for several helpful suggestions in connection with the examples in this section.

Moreover, for purposes of mathematical analysis we require that  $k$  and  $f$  be periodic functions with period  $2\pi$ .

Our problem now is to use (4.1) and (4.2) to find the density  $r(x)$  of the asymptotic response distribution by integrating the derivative of (3.4), namely,

$$(4.3) \quad r(x) = \int_0^{2\pi} k(x;y)f(y)dy = \int_0^{2\pi} k(x-y)f(y)dy .$$

The problem is more complex than the simple form of the functions  $k$  and  $f$  indicate, for the limits of integration must be carefully adjusted. What we are led to is a division of the circle into six regions with a different integrand for each. To begin with, since  $k$  and  $f$  are periodic with period  $2\pi$ , we may replace  $x-y$  by  $t$  and have instead of (4.3),

$$(4.4) \quad r(x) = \int_0^{2\pi} k(t)f(x-t)dt .$$

But since  $k(t)$  is 0 when  $t^2 > a^2$ , where  $a = (\frac{3}{4})^{1/3}$ , we may for this example replace (4.4) by

$$(4.5) \quad r(x) = \int_{-a}^a k(t)f(x-t)dt .$$

The six regions we have to consider are the following:

$$\begin{array}{ll} 0 < x \leq a & \pi < x \leq \pi+a \\ a < x \leq \pi-a & \pi+a < x \leq 2\pi-a \\ \pi-a < x \leq \pi & 2\pi-a < x \leq 2\pi \end{array} .$$



For the first one,  $0 < x \leq a$ , we have to consider two cases:  $t \geq x$  and  $t < x$ . Thus we have, for this region:

$$(4.6) \quad r(x) = \int_x^a k(t)f(x-t)dt + \int_{-a}^x k(t)f(x-t)dt .$$

For both integrands of (4.6),  $k(t) = a^2 - t^2$ , but in the case of  $f(x-t)$ , for the first one  $t > x$  and therefore the argument  $x-t$  is negative, which is not within the domain of definition of (4.2), but given that  $f$  has period  $2\pi$ , we have

$$f(x-t) = f(2\pi + x-t)$$

and  $\pi < 2\pi + x-t < 2\pi$ , whence from (4.2)

$$\begin{aligned} f(x-t) &= \frac{2}{\pi} - \frac{(2\pi + x-t)}{\pi^2} \\ &= \frac{t-x}{\pi^2} . \end{aligned}$$

For the second integrand of (4.6),  $f(x-t) = \frac{x-t}{\pi^2}$ . Therefore, we obtain, for  $0 < x \leq a$ ,

$$\begin{aligned} (4.7) \quad r(x) &= \frac{1}{\pi^2} \int_x^a (a^2 - t^2)(t-x)dt + \frac{1}{\pi^2} \int_{-a}^x (a^2 - t^2)(x-t)dt \\ &= \frac{1}{6\pi^2} (3a^4 + 6a^2x^2 - x^4) . \end{aligned}$$

We turn now to the second region,  $a < x \leq \pi - a$  . Here  $0 < x - t \leq \pi$  and thus

$$f(x - t) = \frac{x - t}{\pi^2} .$$

Hence, for this region

$$(4.8) \quad r(x) = \frac{1}{\pi^2} \int_{-a}^a (a^2 - t^2)(x - t)dt$$

$$= \frac{4}{3\pi^2} a^3 x .$$

For the third region,  $\pi - a < x \leq \pi$  , complications once again ensue. There are two cases to consider:  $t \geq x - \pi$  and  $t < x - \pi$  . If  $t \geq x - \pi$  then  $x - t < \pi$  and  $f(x - t) = (x - t)/\pi^2$  . If  $t < x - \pi$  , then  $x - t > \pi$  and

$$f(x - t) = \frac{2}{\pi} - \frac{x - t}{\pi^2} .$$

Thus, for  $\pi - a < x \leq \pi$  ,

$$(4.9) \quad r(x) = \frac{1}{\pi^2} \int_{x-\pi}^a (a^2 - t^2)(x - t)dt + \int_{-a}^{x-\pi} (a^2 - t^2)\left(\frac{2}{\pi} - \frac{x - t}{\pi^2}\right)dt .$$

After integrating and simplifying, we get from (4.9):

$$(4.10) \quad r(x) = \frac{1}{\pi^2} \left[ \frac{x^4}{6} - \frac{2\pi}{3} x^3 + (\pi^2 - a^2)x^2 \right.$$

$$\left. + (2a^2\pi - \frac{2}{3}\pi^3)x + \frac{\pi^4}{6} - a^2\pi^2 + \frac{4}{3}\pi a^3 - \frac{a^4}{4} \right]$$

for  $\pi - a < x \leq \pi$  .

Because  $r(x)$  is symmetric about  $\pi$  , the other three regions have similar densities. More exactly, the interval  $(\pi, \pi+a]$  corresponds to  $(\pi-a, \pi]$  , the interval  $(\pi+a, 2\pi-a]$  to  $(a, \pi-a]$  , and the interval  $(2\pi-a, 2\pi]$  to  $(0, a]$  . The graph of  $r(x)$  from 0 to  $\pi$  is given in Figure 1 below.

Example 2. Trigonometric smearing distribution. We take as the density of the smearing distribution the cosine function:

$$(4.11) \quad k(x-y) = \frac{1}{4} \cos \frac{x-y}{2} \quad \text{for } -\pi < x-y < \pi .$$

For comparative purposes we retain (4.2) as the reinforcement distribution. Because of the periodicity of  $k$  and  $f$  , we may, as is convenient for this example, replace (4.4) by

$$(4.12) \quad r(x) = \int_{-\pi}^{\pi} k(t)f(x-t)dt .$$

To integrate (4.12), using (4.2) and (4.11), we must only decide how to break up the interval  $(-\pi, \pi)$  according to the branch of  $f(y)$  we are using. We may choose as our subintervals  $(-\pi, x-\pi)$ ,  $(x-\pi, x)$ ,  $(x, \pi)$ , and consider only  $x$  , for  $0 < x < \pi$  .

For the first, we have then:

$$-\pi < t < x - \pi ,$$

whence, multiplying by  $-1$  ,

$$\pi > -t > \pi - x ,$$

and now adding  $x$  ,

$$\pi + x > x - t > \pi .$$

Thus for this interval,

$$f(x - t) = \frac{2}{\pi} - \frac{x-t}{\pi^2} .$$

For the interval  $(x-\pi, x)$  , we have:

$$x - \pi < t < x ,$$

and consequently by the same argument as above,

$$\pi - x > x - t > 0 .$$

Thus for this interval,

$$f(x - t) = \frac{x-t}{\pi^2} .$$

Finally, for the third interval

$$x < t < \pi ,$$

whence

$$0 > x - t > x - \pi .$$

Here  $x-t$  is negative, and we add  $2\pi$  to obtain:

$$2\pi > 2\pi + x-t > \pi+x .$$

Hence for this interval,

$$\begin{aligned} f(x-t) &= \frac{2}{\pi} - \frac{2\pi + x-t}{\pi^2} \\ &= \frac{t-x}{\pi^2} . \end{aligned}$$

Applying these three results for  $f$  to (4.12), we obtain for the asymptotic response density for  $0 < x < \pi$ :

$$\begin{aligned} (4.13) \quad r(x) &= \int_{-\pi}^{x-\pi} \left(\frac{1}{4} \cos \frac{t}{2}\right) \left(\frac{2}{\pi} - \frac{x-t}{\pi^2}\right) dt + \int_{x-\pi}^x \left(\frac{1}{4} \cos \frac{t}{2}\right) \left(\frac{x-t}{\pi^2}\right) dt \\ &\quad + \int_x^{\pi} \left(\frac{1}{4} \cos \frac{t}{2}\right) \left(\frac{t-x}{\pi^2}\right) dt = \frac{1}{2} \left(2 \sin \frac{x}{2} - 2 \cos \frac{x}{2} + \pi - x\right) . \end{aligned}$$

Example 3. Exponential smearing distribution. For a final example, we take as the density of the smearing distribution the exponential function:

$$(4.14) \quad k(x-y) = \frac{e^{-|x-y|}}{2(1-e^{-\pi})} \quad \text{for } -\pi < x-y < \pi .$$

Again we use (4.2) for the reinforcement distribution. The divisions of the interval  $(-\pi, \pi)$  in (4.13) are appropriate here, but in addition we must also divide the interval at  $0$ , since

$$e^{-|t|} = \begin{cases} e^{-t} & \text{for } t \geq 0 \\ e^t & \text{for } t < 0 \end{cases} .$$

We thus have for the asymptotic response density function, for  $0 < x < \pi$  :

$$\begin{aligned} (4.15) \quad r(x) &= \frac{1}{2\pi^2(1-e^{-\pi})} \left[ \int_{-\pi}^{x-\pi} e^t(2\pi+t-x)dt + \int_{x-\pi}^0 e^t(x-t)dt \right. \\ &\quad \left. + \int_0^x e^{-t}(x-t)dt + \int_x^\pi e^{-t}(t-x)dt \right] \\ &= \frac{-e^{x-\pi} + (x-\pi)e^{-\pi} + e^{-x} + x}{\pi^2(1-e^{-\pi})} . \end{aligned}$$

The linear density function  $f$  of the reinforcement distribution and the three asymptotic response densities  $r_1$ ,  $r_2$ , and  $r_3$  of the three examples just given are plotted in Figure 1. ( $r_1$  is for the parabolic example,  $r_2$  for the trigonometric, and  $r_3$  for the exponential.) The differences between  $r_1$ ,  $r_2$ , and  $r_3$  are sharp enough to expect experimental resolution of their relative validity. Actually, one of the most likely candidates for the smearing distribution is a truncated normal distribution with mean zero and variance to be estimated from the data. Such an example was not given here because the integrals cannot be given in closed form.

These three examples bring out the most striking difference between the finite and the continuous non-contingent model: The asymptotic response distribution is not in general identical with the reinforcement distribution;

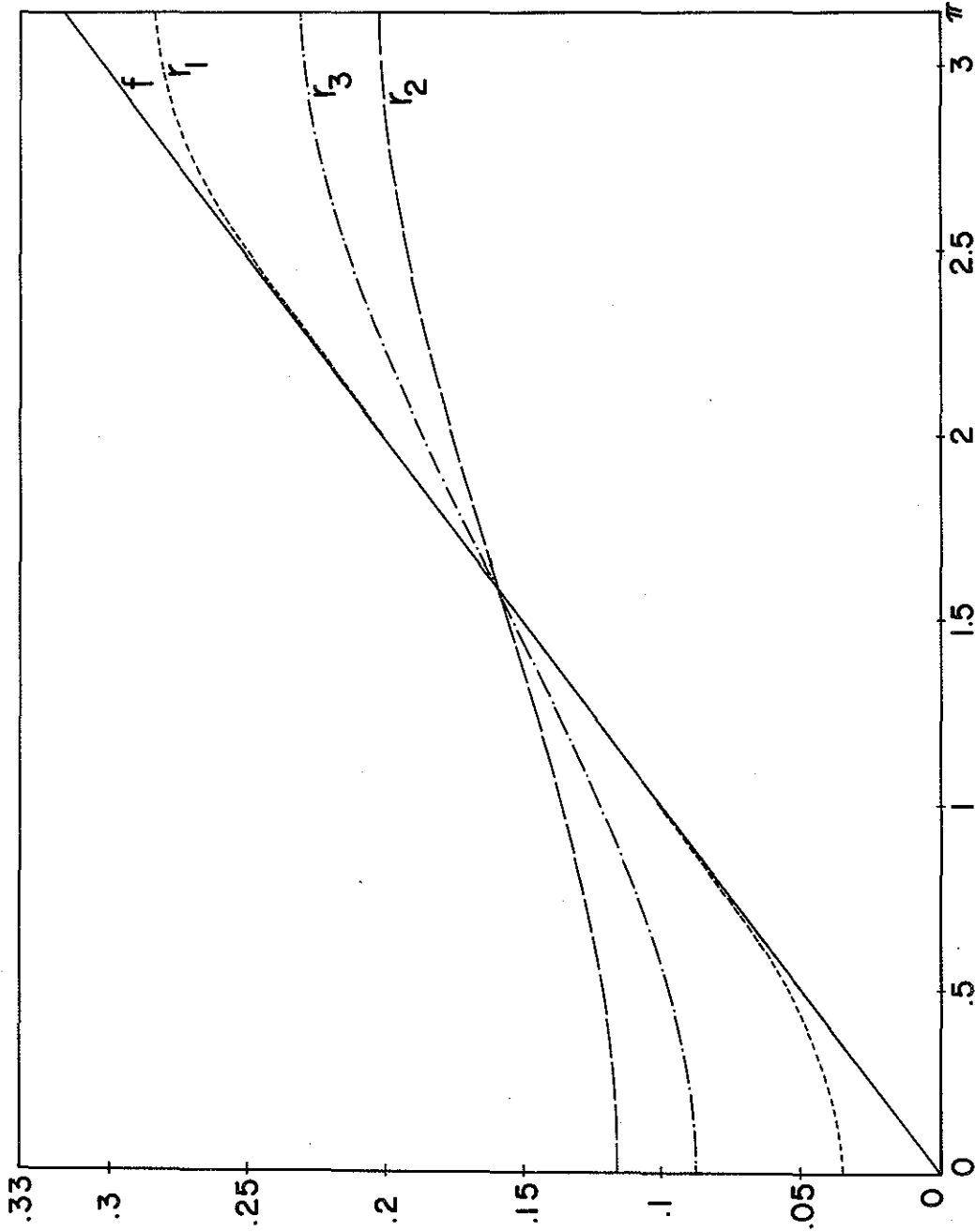


Figure 1.- Asymptotic Response Distribution for Linear Reinforcement and Three Different Smearing Distributions

the obvious analogue of (1.6) does not hold. It is to be anticipated that the exact form of the smearing distribution, and thus the exact relation between the reinforcement distribution and the asymptotic response distribution, will vary from one piece of apparatus to another. For example, in the case of the circular disc apparatus, it seems likely that when measurements are made in radians, the variance of the smearing distribution will vary inversely with the diameter of the disc.

5. Further Results for the Non-contingent Case.

For the non-contingent case of the continuous model we may derive without much difficulty results analogous to those in Section 8 of ES. To begin with, analogous to the moments (5.2) of ES, for a subinterval  $(a_1, a_2)$  of  $[a, b]$ , we have:

$$(5.1) \quad v_{a_1, a_2, n}^v = \int_{a_1}^{a_2} \int_{s_{n-1}} j_n^v(x | s_{n-1}) j(s_{n-1}) dx ds_{n-1},$$

where integration over  $s_{n-1}$  means integration over the whole interval  $[a, b]$  for each of the  $2(n-1)$  random variables corresponding to the sequence  $s_{n-1}$ . Note that the subscript  $n$  is omitted on the second density  $j$ , a practice which is followed in the sequel when it is not needed for clarity.

A general recursion for the moments just defined, analogous to (8.2) of ES, is obtainable, but because of its length, we restrict ourselves here to  $v=2$ , which corresponds to (8.6) of ES. We emphasize again that these results are for the non-contingent case, that is, when for all  $n$



$$(5.2) \quad f_n(y) = f(y) \quad .$$

Theorem.

$$(5.3) \quad V_{a_1, a_2, n+1}^2 = (1-\theta)^2 V_{a_1, a_2, n}^2 + 2\theta(1-\theta) \int_{a_1}^{a_2} \int_a^b r_n(x)k(x;y)f(y)dx dy \\ + \theta^2 \int_{a_1}^{a_2} \int_a^b k^2(x;y)f(y)dx dy \quad .$$

Proof: By virtue of (5.1)

$$V_{a_1, a_2, n+1}^2 = \int_{a_1}^{a_2} \int_{s_n} j_{n+1}^2(x | s_n) j(s_n) dx ds_n \quad ,$$

but each  $s_n = (y_n, x_n, s_{n-1})$  , whence

$$V_{a_1, a_2, n+1}^2 = \int_{a_1}^{a_2} \int_a^b \int_a^b \int_{s_{n-1}} j_{n+1}^2(x | y_n, x_n, s_{n-1}) j(y_n, x_n, s_{n-1}) dx dy_n dx_n ds_{n-1} \quad .$$

Applying Axiom C3 to the right-hand side of the above equation

$$V_{a_1, a_2, n+1}^2 = \int_{a_1}^{a_2} \int_a^b \int_a^b \int_{s_{n-1}} [(1-\theta)j_n(x | s_{n-1}) + \theta k(x; y_n)]^2 j(y_n, x_n, s_{n-1}) dx dy_n dx_n ds_{n-1} \\ = \int_{a_1}^{a_2} \int_{s_{n-1}} (1-\theta)^2 j_n^2(x | s_{n-1}) j(s_{n-1}) dx ds_{n-1} \\ + 2\theta(1-\theta) \int_{a_1}^{a_2} \int_a^b \int_{s_{n-1}} j_n(x | s_{n-1}) k(x; y_n) f(y_n) j(s_{n-1}) dx dy_n ds_{n-1} \\ + \theta^2 \int_{a_1}^{a_2} \int_a^b k^2(x; y) f(y) dx dy \quad ,$$

and the theorem easily follows from this last equation, since the first term on the right is simply  $(1-\theta)^2 v_{a_1, a_2, n}^2$ , and we may integrate out  $s_{n-1}$  in the second term on the right and thus replace  $j_n(x | s_{n-1})j(s_{n-1})$  in the integrand by the mean response density  $r_n(x)$ . Q.E.D.

Asymptotically, we have from (5.3), corresponding to Corollary 8.8 of ES,

$$(5.4) \quad v_{a_1, a_2}^2 = \frac{2(1-\theta)}{2-\theta} \int_{a_1}^{a_2} \int_a^b r(x)k(x;y)f(y)dx dy \\ + \frac{\theta}{2-\theta} \int_{a_1}^{a_2} \int_a^b k^2(x;y)f(y)dx dy .$$

To obtain an expression corresponding to the doublet term  $P(A_{1, n+1} \cap A_{1, n})$ , we need first to define some cross-moments similar to the moments  $v_{a_1, a_2, n}^v$ . Note that the subscripts of  $x$  in this definition do not refer to trial numbers but index the  $v$  variables required for the definition:

$$(5.5) \quad w_{a_1, a_2, n}^v = \int_{a_1}^{a_2} \dots \int_{a_1}^{a_2} \int_{s_{n-1}} j_n(x_1 | s_{n-1}) \dots j_n(x_v | s_{n-1}) j(s_{n-1}) dx_1 \dots dx_v ds_{n-1} .$$

Using the notation  $j_{n+1, n}(x', x)$  for the joint distribution of responses on trials  $n+1$  and  $n$ , we then have:

Theorem. For  $a \leq a_1 < a_2 \leq b$

$$(5.6) \quad \int_{a_1}^{a_2} \int_{a_1}^{a_2} j_{n+1, n}(x', x) dx' dx = (1-\theta) w_{a_1, a_2, n}^2 + \theta [R_n(a_2) - R_n(a_1)] \cdot \\ \int_{a_1}^{a_2} \int_a^b k(x;y)f(y)dx dy .$$

Proof:

$$\int_{a_1}^{a_2} \int_{a_1}^{a_2} j_{n+1,n}(x',x) dx' dx = \int_{a_1}^{a_2} \int_a^b \int_{a_1}^{a_2} \int_{s_{n-1}} j_{n+1}(x'|y,x,s_{n-1}) f(y) j_n(x|s_{n-1}) j(s_{n-1}) \cdot dx' dy dx ds_{n-1}$$

where we make use on the right-hand side of the non-contingency of reinforcement.

Applying now Axiom C3, we have that the right hand is:

$$= \int_{a_1}^{a_2} \int_a^b \int_{a_1}^{a_2} \int_{s_{n-1}} [(1-\theta)j_n(x'|s_{n-1}) + \theta k(x';y)] f(y) j_n(x|s_{n-1}) j(s_{n-1}) dx' dy dx ds_{n-1}.$$

Using now (5.5), and integrating out  $s_{n-1}$  in the second term, we get:

$$= (1-\theta)W_{a_1, a_2, n}^2 + \theta \int_{a_1}^{a_2} \int_a^b \int_{a_1}^{a_2} k(x';y) f(y) r_n(x) dx' dy dx.$$

Finally, integrating out  $x$  in the second term, we obtain the desired result.

Q.E.D.

The similarity of (5.6) to the corresponding expression:

$$(1-\theta)V_n^2 + \theta\pi V_n^1$$

in ES should be emphasized.  $W_{a_1, a_2, n}^2$  corresponds to the second moment  $V_n^2$ ; the probability  $R_n(a_2) - R_n(a_1)$  of responding in the interval  $(a_2, a_1)$  to the first moment  $V_n^1$ ; and the integral to the probability  $\pi$ .

Using the techniques of proof for the preceding two theorems it is not difficult, though somewhat tedious, to get analogues of (8.15), (8.17) --

the covariance term, and the serial correlation  $r_m$  of ES. But since these analogues involve rather lengthy integral expressions in terms of the unspecified smearing and reinforcement distributions, we omit them here.

In conclusion, we obtain some relatively simple expressions for the expected value and variance (asymptotically) of the response random variable  $A_n$ . The expectation of  $A_n$  is defined by:

$$E(A_n) = \int_a^b x r_n(x) dx .$$

Applying the usual techniques to the right-hand side and using Axiom C3, we have:

$$\begin{aligned} \int_a^b x r_n(x) dx &= \int_a^b \int_a^b \int_a^b \int_{s_{n-2}}^b x j_n(x | y_{n-1}, x_{n-1}, s_{n-2}) j(y_{n-1}, x_{n-1}, s_{n-2}) dx dy_{n-1} dx_{n-1} ds_{n-2} \\ &= \int_a^b \int_a^b \int_{s_{n-2}}^b x [(1-\theta) j_{n-1}(x | s_{n-2}) + \theta k(x; y_{n-1})] j(y_{n-1}, s_{n-2}) dx dy_{n-1} ds_{n-2} \\ &= (1-\theta) \int_a^b x r_{n-1}(x) dx + \int_a^b \int_a^b x k(x; y) f(y) dx dy . \end{aligned}$$

Now the first term on the right of the last expression is simply  $E(A_{n-1})$ , and thus we have the recursion:

Theorem.

$$(5.7) \quad E(A_n) = (1-\theta)E(A_{n-1}) + \theta \int_a^b \int_a^b x k(x; y) f(y) dx dy .$$

Moreover, asymptotically

$$(5.8) \quad E(A_{\infty}) = \lim_{n \rightarrow \infty} E(A_n) = \int_a^b \int_a^b xk(x;y)f(y)dx dy .$$

It is not difficult to show that for the three numerical examples considered in the preceding section,  $E(A_{\infty}) = \pi$  . (Note that  $\pi$  is not a probability but the numerical constant 3.1416. .)

The recurrence relation for the variance of  $A_n$  is not simple, but asymptotically we have:

$$(5.9) \quad \text{var}(A_{\infty}) = \int_a^b \int_a^b x^2 k(x;y)f(y)dx dy - \left[ \int_a^b \int_a^b xk(x;y)f(y)dx dy \right]^2 .$$

If it is assumed, for the circular disc or similar apparatus, that the smearing distribution is a truncated normal distribution with mean zero and unknown variance, then (5.8) may be used to estimate (somewhat inefficiently) this variance, for  $\text{var}(A_{\infty})$  may be computed directly from the empirical histogram of asymptotic response frequencies.

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