MEASUREMENT: PROBLEMS OF THEORY AND APPLICATION

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TECHNICAL REPORT NO. 147

June 12, 1969

PSYCHOLOGY SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

Measurement: Problems of Theory and Application[1]

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Those of you who have just heard and enjoyed the fascinating lecture by Professor Blakers may have a certain deja vu sense as you hear what I am going to say. This will be true especially of those who also attended Mr. Mitchell's lecture earlier. To reduce this deja vu feeling, I shall deal initially with questions of measurement that have systematic or formal interest, but that diverge rather sharply from the mainstream of mathematics. From the standpoint of applying the theory of measurement in the social sciences, or indeed even in the physical sciences, many conceptually unsatisfactory things arise about a theory of measurement that is related directly to standard structures in mathematics. The structures widely studied in mathematics are almost without exception infinitistic, and they are almost without exception error-free. In concentrating on the problems of finitude and error, I shall organize what I have to say under four headings.

First, I want to formulate a general viewpoint that is much in agreement with that expressed by Professor Blakers. Second, I shall discuss necessary and sufficient conditions for the existence of numerical measures on finite structures. Third, I shall talk about algebraic theory of error, and finally, I shall comment on nonalgebraic error theory, with some remarks about linear regression and the structural models discussed by Professor Williams.

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[1]This paper is based on a lecture given at a UNESCO Seminar at the University of Syndey, Australia on May 24, 1968. The content of the paper reflects research that has been supported by the National Science Foundation.

1. General Viewpoint

We begin with measurement procedures that are fundamental in the sense that they assume no prior numerical results. We may represent the empirical situation in most cases by a set A that is the set of objects or phenomena under consideration, and a finite sequence of finitary relations R_i on this set. Such an object, $\mathfrak{A} = < A, R_1, \ldots, R_n >$, is ordinarily called a relational structure. I deliberately have chosen <u>relations</u> rather than operations to represent the experimental or measurement procedures of a fundamental character, because when we consider actual procedures, the closure properties so characteristic of ordinary mathematical operations clearly lead us into infinistic idealizations that are not happy idealizations for many applications of measurement. For example, if we impose a closure condition on our algebra of operations for the measurement of length, then we are committed almost at once to postulating the existence of lengths of arbitrarily great size. If we use relations instead of operations, no such commitment is required, and we can restrict considerations entirely to finite sets and finite relational structures. (By a <u>finite</u> relational structure, I mean a relational structure in which the basic set A of objects is finite.)

In this general viewpoint, two formal problems should be solved for any fundamental measurement procedure or fundamental theory of measurement. From a formal or mathematical standpoint, we characterize the class of relational structures that satisfy the empirical procedure or the theory by stating the axioms that must be satisfied by each structure. The first formal problem then is to prove a representation theorem for any structure satisfying the axioms. Ordinarily in order to call the theory a theory of measurement, this representation theorem should show that any structure satisfying the axioms may be mapped homomorphically into the real numbers. The restriction to the real numbers is not critical. Mapping into a structure closely related to the real numbers is. For example, in the case of multidimensional scaling, what is desired is a mapping into an n-dimensional Euclidean space rather than the set of real numbers themselves. One point about the representation theorem needs clarification. When the structures are finite there is no problem of mapping them homomorphically into the real numbers with some arbitrary relations on the real numbers. The interest of the problem is rather to provide in advance the numerical relations in terms of which the numerical structures should be defined. The homomorphism then must be relative to these given numerical relations.

The second formal problem is that of uniqueness. How unique is the homomorphism mapping a given structure into the real numbers? In the classical measurement theory of mass or distance, for example, we expect the mapping to be unique up to a positive similarity transformation.

This way of looking at theories of measurement is not special in any sense to the domain of measurement procedures. In all areas of mathematics, it is standard to search for representation theorems for structures of primary interest and also to ask about the uniqueness of the representations obtained. Professor Blakers has already mentioned the familiar classical example, namely, the representation theorem for plane geometry in terms of Cartesian coordinates, and the proof that this representation is unique up to the group of Euclidean motions. In the context of the present discussion, it is perhaps worth mentioning that classical synthetic geometry can be axiomatized in an elementary fashion as a relational structure. The basic set A is the set of points, and the two relations we define on the set A are the ternary relation of betweenness between points and the quaternary relation of equidistance. The relation of equidistance means that the distance between points x and y is the same as the distance between points u and v.

Still another example of considerable physical interest is to be found in the relation between measurement and the theory of special relativity. In this case, greater interest has been attached to the uniqueness theorem than to the representation theorem. We may show, for example, that preservation of the relativistic length of inertial segments is the only assumption needed to prove that any two inertial frames are related by Lorentz transformations (Suppes 1959). More recently, Zeeman (1964) has shown that by using slightly stronger assumptions about the number of dimensions $(n \geq 3)$, it is possible to postulate that the time-like partial ordering of points is preserved in order to obtain the Lorentz transformations. From the standpoint of the theory of measurement, it is interesting to find that no additional physics is required to derive the Lorentz transformations.

I mention these examples of geometry and relativity, because there has been a recent tendency for the literature on the theory of measurement to become isolated from other domains of science, and I regard this as unfortunate in view of the close connections of the sort just described.

2. Necessary and Sufficient Conditions

If we undertake to formulate a fundamental theory of measurement, we should recognize that it is important and interesting to state axioms that are not only sufficient, but also necessary. Such axioms give us a better sense of understanding the nature of the theory. There also are more practical reasons of application for seeking necessary and sufficient conditions. In considering collections of objects of a similar sort whose properties are to be measured, we do not want to restrict the collections of objects because of extraneous existential assumptions. We naturally ask what minimal conditions can we assume guarantee the existence of a measure? Existential conditions, for example, that are sufficient but not necessary unduly restrict the range of structures that fall within the theory. To seek necessary and sufficient conditions is to seek a bare-bones characterization of all the structures that intuitively should be brought within the framework of the theory.

At this point it may be useful to look at a simple example. Consider binary structures, that is, structures consisting of a set A and a binary relation R on this set. It is natural to ask what necessary and sufficient conditions exist that make a binary structure homomorphic to a numerical structure $\mathfrak{N} = \langle N, < \rangle$ where N is a set of real numbers and $<$ is the usual numerical relation <u>less than</u> restricted to N.

If the set A is finite, the answer is simple. The relation R must just be asymmetric, transitive, and connected on A. That is, the following three axioms must be satisfied in the structure. For every x, y, and z in A:

<u>Axiom</u> 1. <u>If</u> xRy <u>then</u> <u>not</u> yRx.

<u>Axiom</u> 2. <u>If</u> xRy <u>and</u> yRz <u>then</u> xRz.

<u>Axiom</u> 3. <u>If</u> $x \neq y$ <u>then</u> xRy <u>or</u> yRx.

In the finite case, the proof of the necessity and the sufficiency is obvious and need not be discussed further.

If we relax the restriction that the set A be finite, then the three axioms are no longer sufficient, but only necessary. To see that the axioms are not sufficient, we observe that they have models of arbitrary high cardinality. When a relation R satisfies these three axioms, however, any homomorphism also must be an isomorphism, but there cannot be a one-one function imbedding models of arbitrarily large infinite cardinality into the real numbers. To imbed such an ordering in the real numbers, we must add an additional condition. In the present case a relatively simple answer is at hand. An ordering that satisfies the above three axioms also must have a countable order-dense subset in case the set A is infinite. However, this condition for the infinite case is not really interesting from the standpoint of the theory of measurement.

Because the necessary and sufficient conditions are so obvious and simple for finite orderings, we initially might expect the situation to be the same or close to the same for relational structures that seem only slightly more complex than orderings. The next simple class to consider is that of orderings on differences as well as orderings on the objects themselves. For example, in a psychological experiment, subjects might be asked to judge whether tone x is closer in pitch to y than tone u is to v. In other words, we ask the subject to make judgments about differences as well as order. More generally, we can think of asking for judgments of relative similarity, that is, the judgment that x is at least as similar to y as u is to v. The real-valued mapping f we want for such a quaternary relation D is this:

(1) xyDuv if and only if $f(x) - f(y) \leq f(u) - f(v)$.

Keeping this representation in mind, we shall call a relational structure consisting of a set A and a quaternary relation D that satisfies this condition a <u>difference structure</u>. The problem is to find the elementary necessary and sufficient conditions that a difference structure must satisfy to guarantee the existence of such a measurement function f.

As in the case of orderings, let us restrict ourselves to finite sets and ask what sort of necessary and sufficient conditions we would like to find. The simple thing about the three axioms on orderings is that we can check a binary structure to see whether it satisfies the axioms by looking at no more than triples of objects. To check on asymmetry and connectedness, only pairs of objects need be examined. To check on transitivity, triples of objects need inspection to determine whether there are any intransitive triads.

We would expect the situation to be somewhat more complex for difference structures, but it still would be valuable as a first step to seek generalizations of transitivity. For example, transitivity of differences, which would require six variables for expression, would demand a check on sextuples of objects in the set A to see if the necessary and sufficient conditions for the existence of a measure could be stated in a relatively simple way, especially in a way that could be checked either by hand or with a simple computer program. It would be desirable to have an upper bound on the size of the n-tuple needed to check on the existence of a measure. An upper bound independent of the cardinality of A shows that the structures for which a measure exists do not get essentially more complicated as the cardinality of the set A increases. If we eliminate as axioms existential assertions, then Dana Scott and I showed some years ago (1958) that for open sentences or universal sentences, that is, for sentences that use only universal quantifiers standing in front, it is not possible to give a finite list of necessary and sufficient conditions that any finite structure must satisfy in order to be a difference structure. In a subsequent paper, Scott (1964) gave necessary and sufficient conditions in terms of an infinite schema that increases in complexity as the number of objects in the set A increases and that in terms of elementary open sentences is equivalent to a countable infinity of such sentences. What is especially important about this condition is that it requires checks on arbitrarily large n-tuples of the number of objects when the set A increases.

To those interested in multidimensional scaling, it is worth remarking that the results Scott and I obtained have been generalized recently to n dimensions by Titiev (1969). He shows that if we attempt to represent similarity judgments in an n-dimensional space with an Euclidean metric, then again necessary and sufficient conditions in terms of a finite list of open sentences cannot be given. Titiev also establishes a similar result for additive conjoint measurement in n dimensions.

Perhaps the best necessary and sufficient conditions yet found in a single paper are in Scott (1964), but his examination of other cases, including measurement of subjective probabilities or measurements of mass or distance, shows that a simple finite list of necessary and sufficient conditions cannot be found in any of the standard cases. In fact, it is an interesting problem to ask for what cases other than those of simple ordering can necessary and sufficient conditions be found for finite domains? A closely related example is given in the next section, but I must confess that I cannot think of any other example that is of genuine interest.

One direction for further investigation in the literature is the tightening of requirements on necessary and sufficient conditions. For instance, by requiring that the measure assigned to the finite structure be unique up to some classical group of transformations like the group of linear or similarity transformations, we could increase the requirements for necessary and sufficient conditions and in this way find simpler solutions in some cases. I cannot, however, at this time report any non-trivial positive results in this direction.

3. Algebraic Theory of Error

The easiest place to begin looking at the problem of error is in simple orderings. We may ask how the problem of error can be introduced without going to probabilistic considerations. In other words, what can we say at the algebraic level about a theory of error combined with the theory of order? It is pleasant to report that in this case a simple solution is at hand. Surprisingly, the idea was not introduced in an explicit way much earlier in the literature. The concept of a semiorder, combining the ideas of error and order, was introduced first by Luce (1956), and the axioms were simplified later by Scott and me (1958). Axioms are stated for binary structures $\mathfrak{A} = \langle A, P \rangle$, where the intended interpretation of the relation P is that of strict precedence or strict preference. The axioms are just the following three for any x, y, u, and v in A:

Axiom 1. *Not* xPx .

Axiom 2. *If* xPy *and* uPv *then either* xPv *or* uPy .

Axiom 3. *If* xPy *and* uPv *then either* wPy *or* uPw .

The intuitive idea of the axioms in the representation is that objects not in the relation of strict precedence fall below a threshold of discrimination. The theorem that can be proved is that in the finite case the relation P may be represented as follows:

(2) xPy if and only if $f(x) > f(y) + \epsilon$, with $\epsilon > 0$.

In reference to our earlier discussion, it also is pleasing to report that these three axioms are necessary and sufficient for the kind of representation just shown when the set A is finite.

A natural part of the algebraic theory of error is to disregard considerations of order and to consider only the relation between objects not discriminable, that is, objects that lie within the threshold of discrimination. Let I stand for such a relation of indiscriminability. It is apparent that the relation I should be reflexive and symmetric--the following two axioms are satisfied for any two objects x and y in the basic set A of the structure:

Axiom 1. xIx .

Axiom 2. *If* xIy *then* yIx .

It is also clear what our intended representation for the set I is. We want to find a representation of the following sort:

(3) xIy if and only if $|f(x) - f(y)| \leq \epsilon$ with $\epsilon > 0$.

One would expect that similar simple necessary and sufficient conditions for the relation I could be found as in the case of the relation of strict precedence given above. Once again, however, the answer is negative. This recently was proved by Fred Roberts (1968) in his Stanford dissertation. The result is similar to the one obtained earlier for difference structures. No finite set of elementary open sentences can characterize the necessary and sufficient conditions to obtain the representation required by (3). In the case of the indiscriminability relation, the heart of the proof lies in the fact that indiscriminability cycles of ever greater size must be excluded. Let a connecting line segment indicate that the objects are indiscriminable. We must exclude cycles of arbitrary size as shown in Figure 1. Excluding

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Insert Figure 1 about here
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such cycles means that in examining specific data on indiscriminability, we cannot check on the existence of a representation function just by looking, as we might like to, at pairs or triples or quadruples of objects. We must check n-tuples of arbitrary size to make sure that a representation exists.

Again, as in Scott's (1964) results for difference structures, Roberts (1968) does give an infinite set of necessary and sufficient conditions for representability of relations of indiscriminability on finite sets. The most important axiom schema is the one that excludes the countable list of cycles of the kind described above.

The axiomatic situation with respect to algebraic theory of error for more complicated measurement structures is not yet entirely satisfactory. Some relatively complicated necessary and sufficient conditions for the measurement of subjective probability with a semiorder replacing

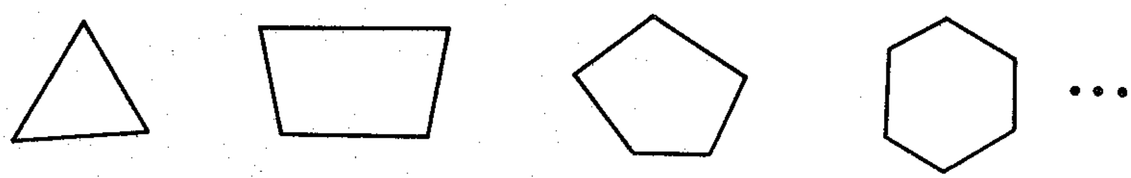

Figure 1. Cycles of indiscriminability.

the ordinary simple ordering are given in the Stanford dissertation of Zoltan Domotor (1969). Earlier nonaxiomatic work on semiorders in extensive measurement can be found in Krantz (1967).

These difficulties of finding a workable formulation cast doubt, it seems to me, upon the practicability of applying algebraic theories of error to real data. In fact, I know of no place in the experimental literature where the ideas I have just described are used in a systematic way. Probabilistic methods or some other approach--for example, the kind of thing that can be got from using linear programming techniques--has been used in all the studies I know about. It seems to me that an important problem for measurement theory is to determine at a deeper level whether there are serious possibilities of actually applying the algebraic theory of error to real experimental data.

4. Nonalgebraic Theory of Error

Standing apart from fundamental theories of error is a very substantial applied theory about the analysis of error in measurement. It is not possible here, in the short space that remains, to review this literature, but certain parts of it are so close to problems of measurement in the social sciences that I would be remiss not to mention their connection to the topics I have already dwelt upon. Of the normal family of general methods, none is more common than that of regression, and the brief remarks I make here can be restricted to regression models without any loss. The first thing to note about such models is the absence of an axiomatic basis in the subject matter of the phenomena being measured. As Professor Williams rightly emphasized in his lecture earlier in the Seminar, the appropriate use of regression models is in the exploration of an area and in the identification of relevant variables. The framework of regression itself does not provide a natural setting in which to investigate the properties of the structures generic to the domain. As he also emphasized, we desire to pass from regression models to structural models that postulate more about the mutual relationships holding in the given domain. The widespread use of regression models in every area of the social sciences bears witness to two things--first, to the superficiality as yet of much of our theorizing, and second, to the absolute necessity of taking account of errors in the relationships we study. It typically is the case in a regression analysis that the error term does not really refer to errors in the measurement of the variables but rather to the inability of the interrelationship among the independent variables to account for variations in the dependent variable.

Still another way to state the matter is this. We can hope to find underlying structural models that will justify, at least as a first approximation, the regression models that are so easily applied to the study of phenomena in almost every domain. I would like to stress the subtlety of the relation that can exist between structural models and regression models by mentioning very briefly some of my own work. We recently have begun to apply probabilistic automata models to the analysis of performance on arithmetical tasks by young children. These probabilistic automata constitute structural models, I believe, in almost anyone's sense of the term. We attempt to give a detailed processing account of steps the students execute in applying an algorithm for finding a numerical answer. The probabilistic aspects of the automaton are adjusted to data to fit the errors made by students. Of course, if students made no errors whatsoever, the experimental study would be trivialized, and we could represent their behavior by a simple finite deterministic automaton. Under a natural set of assumptions about sources of errors, we can pass from the automaton model to actual analysis of the data in terms of a simple regression model by taking the logarithm of the probability of a correct answer.

In a situation like this, the regression model is no longer one used simply to explore relevant variables; it is an important statistical tool in testing a rather elaborate underlying structural model. The theory of error built into the regression model is then directly useful in fitting the automaton model to the experimental data.

I realize that I have said only the most superficial things about nonalgebraic error theory, but it is necessary to bring discussion to an end at this point.

There is one final thing that I would like to say as an expression of my own feelings about measurement. As we explore a new area of science, as we develop new insights into a familiar area, or as we improve our techniques of measurement, we should use those techniques to identify important variables and to move from that identification to assumptions that go beyond the theory of measurement itself. Another way of putting the matter is this. I think that all of us interested in the theory of measurement should keep an eye cocked at all times for the more general conceptual framework within which we are working and try to use the results of measurement to deepen our understanding of that framework, and even on occasion, totally to rebuild it. In the social sciences especially, sound use of the theory of measurement can contribute as much to matters of general theory construction as to the improvement of our empirical methods of investigation and data analysis.

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