ABSTRACT. Difficulties in the measurement of physical quantities arise in classical physics as soon as classical physics has to deal with waves in a concrete sense. The concept of a monochromatic plane wave is completely and precisely defined, but it remains abstract. Concrete waves as collective objects, on the contrary, are burdened with spectral inequalities, characteristic of classical field magnitudes.

Quantum mechanics defines physical magnitudes as operators acting on a vector, the vector of state of a given system. Classical mechanics, for example Lagrange’s analytical mechanics, is completely foreign to this conception. The evolution of a mechanism is completely determined when the values of its generalized coordinates \( q_i \) and of their derivatives with respect to time \( \dot{q}_i \) are given at any time. The state of such a mechanism is nothing less than the assignation of these geometrical magnitudes (distances and angles) and these kinematical magnitudes (generalized velocities).

It is true to say that quantum mechanics has completely transformed both of the concepts of physical magnitude and state. Nevertheless, we must not too hastily give quantum mechanics what belongs to the classical theory of fields.

The proper concepts of physical magnitudes as operators and of states as vectors are relevant to such a theory. Quantum mechanics steps in when Einstein’s and de Broglie’s equations associate properties of particles with properties of waves.

My aim here is to analyze the conceptual change of the concept of physical magnitude within the limits of classical physics. I shall first describe the main characteristics of the field magnitudes. This sketch will then suggest some philosophical conclusions.

The celebrated representations of Nature by the atoms and their composition in Greece and the less well-known analysis of the tides by

Posidonius the Stoic testify that ancient physics had already recognized the necessity of using two conflicting conceptualizations in order to describe matter.

On the other hand reality is conceived as made of discontinuous, localized, exclusive bits which move by transportation according to well-defined trajectories. Physics has essentially to state the rules of conservation that govern the collisions of the atoms and their aggregates. Collisions require laws of conservation relative to physical magnitudes which may be locally defined. But the forces which finally yield their collisions and secure unity for a world act at a distance. Newton showed how to work with this hypothesis of separated centres of gravity. When Newton's laws are rewritten in terms of gravitational fields, this introduction of the field conceptualization means that the atomist uses a formal trick rather than that he makes concessions.

On the other hand there are genuine phenomena of propagation without transportation, where the magnitudes involved are continuous, defined at every point of space, and, therefore, everywhere present, and, moreover, intersect their respective paths without disturbing them. Because it is continuous, this conceptualization is more akin than its rival to the requisites of geometry. This is the reason why the followers of Descartes and Huygens did not give ground despite the successes of Newtonianism in mechanics until their apparently final victory with Young and Fresnel. Compare, for example, the laws of impact of one body against another one with the law of the beat of two sound waves in the air. In the first case, we deduce the laws from the principle of conservation of the quantity of motion, without being able to describe what happens at the instant of impact. By contrast, given two sound sources with slightly different frequencies, we have only to make an algebraic addition at each point in order to construct the figure of the interference in time with all its particulars.

Geometry shows, however, that a lot of surprises may be stored in a completely intuitive construction. Let us review two of them under the head of the categories of causality and substance.

When from a law of conservation a physicist predicts how two billiard balls will rebound, he simply equates two accounts before and after the impact. There is succession, there is not causality. Prediction is still possible, but one should speak of global determinism rather than of strict causal laws, since the balance of observable quantities before and after an event does not imply that the event itself be subjected to
a detailed and continuous computation. Hume’s doubt is to the point. But add two waves. Each development is entirely determined once the amplitude, the rate of temporal evolution (pulsation) and the rate of spatial progression (wave number) are fixed. And their superposition obeys the same complete determination. We follow it as we follow them. Causality is here the product of algebra. The surprise comes, nevertheless, if we draw the physical consequences from the superposition, namely that two vibrations, when added, mutually reinforce or extinguish themselves, according as the addition is made at two troughs or at two crests on the one hand, or at a trough and a crest on the other hand. The astonishing phenomena of interference are thus explained away without introducing Kant’s so-called negative magnitudes.\footnote{Without forgetting that monochromatic waves belong to abstract physics, not to concrete physics, we might say, figuratively, that, putting together two illuminations may produce obscurity. In the same way, it will be seen that all the disturbances bound with a superposition of waves infinitely extended over space may be cancelled everywhere except over the tiny extension of a packet’s centre.} Without forgetting that monochromatic waves belong to abstract physics, not to concrete physics, we might say, figuratively, that, putting together two illuminations may produce obscurity. In the same way, it will be seen that all the disturbances bound with a superposition of waves infinitely extended over space may be cancelled everywhere except over the tiny extension of a packet’s centre.

As to the category of substance, the surprise came off in two episodes. First, in his Analytical Theory of Heat, Fourier analyzed the diffusion of heat without taking sides with the upholders either of the caloric or of the kinetic theory. Whereas the physicists had to understand the mechanisms according to which columns of air are displaced by the propagation of sound in order to state and to solve the wave equation of sound, the stationary distribution of the temperatures that results from the diffusion of heat in a wall or in a ring is obtained though no model in terms of fluid or molecules is offered to support the distribution. Auguste Comte, the founder of positivism, well understood the implications of Fourier’s method: physical magnitudes can be measured and known, while the substances that they make manifest are kept in complete ignorance. Therefore phenomenal magnitudes, not substances are relevant to mathematical physics.

The second episode is more telling still. In Fourier’s perspective, physical magnitudes were abstracted from their substances, but their substances, whatever they might be, were not denied a physical existence. On the contrary, when the mechanical theory of light, as construed by Fresnel (who assimilated the vibrations of light with the transverse elastic vibrations of solids and gave the ether, which was supposed to support them, the paradoxical properties of absolute incompressibility
and solidity) was forsaken and Maxwell’s electromagnetic theory was adopted, it was gradually realized that field magnitudes need no substantial medium to which they should adhere. The last tie between magnitudes and substances was broken.

Field magnitudes are superposable and may live an autonomous existence independent of a material medium. These unexpected physical phenomena result from the geometrical constraint of continuity. A pure analytical surprise, if I dare say such, following Wittgenstein’s ukase, still waits for us, when boundary conditions are placed upon the solutions of the wave equation, as is required by the possibility of experience, for example, a plucked vibrating string, a vibrating membrane, a system of two coupled pendula, or the evolution of a particle in a potential well. Fourier’s trigonometric series decomposes any periodic function which is regular enough into an infinite sum of harmonic functions affected by suitable coefficients that determine the relative participation of each function in the whole. Two analytical phenomena relative to such series claim our attention: the emergence of eigenfunctions and eigenvalues and the coupling of the magnitudes.

Owing to boundary conditions, discontinuity, so to speak, emerges from continuity. When the vibrating string is fixed at two points, the space part of the wave function, which is an ordinary differential equation, is constrained to vibrate at certain natural modes. The wave number, and with it the frequency, is allowed to take as values only integral multiples of a constant function of the period. These values are called eigenvalues. To each of them there corresponds an eigenfunction. The eigenfunctions have two important properties. They are orthogonal and complete. This means that they define a generalized vector space of an infinite number of dimensions and thus afford a convenient basis for developing any regular enough function that satisfies the same boundary conditions.

Understood as a theorem of generalized vector analysis, where the elements of the vector space are the real, continuous functions of a real variable defined over an interval, Fourier’s decomposition has far-reaching consequences for the concept of physical magnitude. In ordinary geometry a length is defined by the coefficients which multiply its projections upon a Cartesian orthonormed basis. In the same way let us suppose that the state of a system is characterized by a given periodic function of a continuous variable. Then, owing to Fourier’s analysis, the spectrum of the Fourier coefficients or eigenvalues, if the
eigenfunctions have been previously normalized, will define the parts of a physical magnitude. As the spectra of eigenvalues and eigenfunctions are produced by the action of certain operators upon the periodic function, physical magnitudes become associated with operators. A simple example is given by the energy theorem. The energy of a wave over a period is proportional to the square of its amplitude over the period. The operator, or rather the functional — since the transformation is from a vector to a number — acting upon the function of time is here the definite integral over the period. According to the theorem, the total energy of a wave is simply the sum of the energies of all Fourier’s components. This is precisely Pythagoras’s relation.

Fourier’s analysis may be extended to complex functions and to non-periodic functions. In the first case provisions are made for getting real eigenvalues: the operators must be Hermitian. In the second case, continuous spectra, integrals and Fourier transforms, respectively, replace discrete spectra, series and Fourier components. The integrals which occur in Fourier transforms are reciprocal functions of continuous variables: the wave vector and the position (since we took the space form of the wave equation). Analytical considerations exhibit a classical relation between the widths of two Fourier transforms. Their product has an inferior bound.

Let $\Delta k$ be the width of the scale of undulation, the inverse of the wavelength, and let $\Delta x$ be the characteristic height or spread of the one-dimensional position, i.e., the space extension containing the centre of the wave packet. The function $\psi(x, 0)$ is obtained by integrating its Fourier transform over $k$. If $\Delta x$ is greater than $1/\Delta k$, the transform oscillates several times within the interval $\Delta k$ and the integral over $k$ takes a negligible value: there is destructive interference. If $x = x_0$, $\Delta x$ being less than $1/\Delta k$, the function which is integrated does not practically oscillate and its integral over $k$ takes an appreciable value; the wave packet centre, where the amplitude $(x, 0)$ is maximal, is situated at $x = x_0$. Therefore, to form a wave packet with a limited extension — to simplify let us consider an abstract wave packet by replacing the concrete superposition of an infinity of plane waves by a discrete interference of such waves, finite in number — we need several harmonic components of wavelengths, the range of which is the more extended the more the wave packet is localized.\(^3\)

The spatial spectral inequality couples the two physical magnitudes: wave vector and position. But since physical magnitudes have been
associated with operators, the question arises: what operators are associated with wave vectors and positions, and what properties of operators are responsible for the coupling of these magnitudes? The relevant operators are shown, respectively, to involve a first derivative with respect to space and a multiplication by the space coordinate. Each of them is characterized by a spectrum of eigenfunctions and eigenvalues. The basis afforded by the system of the eigenfunctions of one of them allows the development of the other. But coupled operators do not admit the same eigenvectors and do not commute. The magnitudes with which they are associated are therefore not simultaneously measurable with an accuracy superior to a finite given quantity.

Analyzing the temporal form of the wave equation would produce similar conclusions. A temporal spectral inequality should take the place of its spatial counterpart. Instead of a relation between the widths of the wave and the position vectors, the relation would be between the width of the pulsation spectrum and the width of the temporal extension.

These limitations have nothing to do with quanta. They obtain within the bounds of classical physics and even within the bounds of the part of classical physics that precisely deals with continuous magnitudes par excellence. Specific laws rule the propagation of collective phenomena, the form of which may be confined while the superposed waves fill the whole space. The interferences act constructively only within a thin volume around the centre of the wave packet and destructively everywhere else. As the group of waves runs towards either circle that limits the perturbation caused by a stone thrown in a pond, this form may get a proper velocity, different from the phase velocity. The velocity of the maximum of the wave packet is not the mean phase velocity, since the component waves in a dispersive medium have different velocities because of their different wavelengths and the interferences slowly change the determination of this maximum. This group velocity depends on what spectral extensions the pulsation and the wave vector of the wave packet have.

Therefore it is no wonder that communication theory, which is entirely grounded on classical principles, presents us with the new concept of physical magnitudes associated with operators, when pulsations and gains are measured by linear transmission of messages.

The mutual relation between signal and transmission channels gives rise to the concept of impedance, a physical magnitude which is generally represented by an operator and is measurable when the signal is an
eigenfunction of this operator, the result of the measurement being the associated eigenvalue.†

II

Both of the basic concepts of classical physics raise difficulties that are bound up with conciliating discontinuity and continuity. A material point means a finite quantity of matter, but without extension and therefore infinitely concentrated. In so far as a wave is infinitely extended and everywhere defined by finite magnitudes, it is not burdened with a problem of singularity. The application of the laws of science is nowhere prohibited. Huygens' light waves, which progress through the void, or rather through the ether, with the same velocity may even be said to be unproblematic. The paradoxes that affect the classical concept of physical magnitudes begin with Young's or Fresnel's waves, with dispersion.

Reading an equation – or rather into an equation – is not an easy task. The solutions of the wave function are circular functions of the difference between two products: of the wave vector into the space coordinate, and of the pulsation into the time coordinate. At the beginning the physicists supposed that the pulsation which is a function of the wave vector always expresses the product of this vector with a constant phase velocity. Abandoning this supposition led to important progress in physical field theory during the nineteenth century.

This progress in a sense answers, or in another sense, questions philosophical positions which go back to Descartes and Huygens on the one hand, and to Kant on the other.

Eager to avoid Newton's singularities Kant introduced, besides ordinary or extensive magnitudes corresponding to quantities and to the axioms of intuition, intended specific magnitudes, called intensive magnitudes and corresponding to qualities and to the anticipations of perception. What he had in mind is explained in a book by him which has been 'unduly neglected' (except by Patrick Suppes), the Metaphysical Foundations of Natural Science, where, in the division of science of motion between kinematics and mechanics, he inserts dynamics. Dynamics relies on the opposition of two forces – attraction and repulsion – the balance of which accounts for the size of the objects in the universe. Let us remark that quantum mechanics will answer the same question by
Putting forward Heisenberg's inequality that results from the coupling of the magnitudes occurring in the wave equation, once it is interpreted in terms of quanta, and by postulating for identical particles a specific behaviour which, in the case of antisymmetry, obeys Fermi's principle of exclusion.

This remark and Kant's own weaknesses of construction advise us to replace Kant's so-called intensive magnitudes by the specifically 'dynamical' concept of a wave, a substitution hinted by Kant himself with respect to colors and musical sounds. This move would give Kant's dynamics its proper scope. All the physical magnitudes should be conceived of as field magnitudes, what remains of corpuscular ideas having to be explained away as figures of speech. Kant often insists on the fact that continuity is built into the possibility of experience: an object, he says, could not be known, were its properties not given as physical magnitudes. But it is precisely those properties that should not meet the requirements of continuity imposed by space and time as forms of possible experience that could not be given as physical magnitudes. The supreme transcendental principle goes so far as to identify the possibility of experience and the possibility of the object of experience.

In other words, waves and their magnitudes do define any possible phenomenal reality.

In this view, when it is said that matter is opposed to radiation as corpuscles are opposed to waves, this opposition results from a sheer appearance, since every physical description must fit in with the conditions of possible experience. Maybe atomistic hypotheses have a heuristic value. They are, however, deprived of ontological import. Phenomena are continuous. The question whether reality in itself is atomic or continuous, or whether the universe is finite or infinite bypasses the power of our knowledge and involves reason in dialectical antinomies.

Such a view was generally accepted until the beginning of this century. Duhem, for example, and with him all the followers of energeticism, though they did not profess either that space and time be the subjective forms of our intuition or that pure reason be affected with insuperable contradictions, agreed that physical theories should only 'save the phenomena' without speculating upon the so-called reality of atoms.

Recalcitrant phenomena compelled the physicists to renounce this kind of phenomenalism, to admit the wave–particle dualism within the facts and therefore the possibility of experience. It is remarkable that the
revolution brought about in our notions of physical magnitude and state originated from the very domain of radiation where the classical developments and difficulties that have been described had arisen. Quantum mechanics may even be constructed from merging the concepts of function and vector, as is required by Fourier's generalized analysis, on the condition that pulsation and wave vector be respectively associated with energy and momentum through Planck's constant.

But let us stay within the bounds of classical physics and ask about the meaning of classical field magnitudes. It is here that a distinction used by Descartes and by all the Cartesians, Malebranche, Huygens, Spinoza and Leibniz will be relevant, namely the distinction between abstract physics and concrete physics. Its best and ideal illustration (since the mathematical and physical developments came later) is probably given by analyzing the concept of wave.

A simple wave is an elementary harmonic wave, i.e., a sinusoidal function which moves all in one piece with a phase velocity that depends on the material medium. What defines the abstract individuality of these elementary waves are the quantities by which the different periodicities are determined (period and pulsation for time, wavelength and wavenumber for space). A successive section shows equal amplitudes each time the variable \( t \) has grown a period and an instantaneous section shows equal amplitudes each time the space variable has grown a wavelength. Such waves belong to abstract physics. According to the temporal spectral inequality, the temporal extension of the sinusoidal wave is infinite with a unique pulsation. But a monochromatic plane wave has no physical reality. Only superpositions of harmonic waves may be realized.

Concrete physics then deals with composed or collective objects. But while abstract individual objects were well defined, difficulties and paradoxes arise when concrete individuals, that is groups, are considered. Monochromatic waves were well defined in so far as one precise value of the pulsation and of the wave number were associated with them. On the other hand they are denied any localization in time and space. The abstract character of such simple individuals directly results from the spectral inequalities. It has no counterpart in classical mechanics of corpuscles where a centre of gravity has at any time a well-determined localization and momentum. For field magnitudes only groups may be localized, but owing to the spectral inequalities, localization is coupled with complementary magnitudes. A system relevant to classical
mechanics does not necessarily vibrate with a definite wavelength or pulsation. Therefore, in so far as continuity prevails, physical reality only belongs to systems which are not susceptible of complete determination, though their internal evolution follows a strictly causal law.

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NOTES

1 Versuch den Begriff der negativen Größen in die Weltweisheit einzuführen, by Johann Jacob Kanter, Königsberg, 1763.
4 In radio-electricity the inequality relation is written:

\[ \tau \cdot \Delta \nu \sim 1 \]

where \( \tau \) is the time interval separating two successive annulments of the signal, and \( \Delta \nu \) is the extension of the signal’s spectrum (Blaquièrè, A.: 1960, Calcul matriciel, Paris: Hachette, II, pp. 123, 124). It is rare, in classical physics, that a magnitude is represented by an operator. See, however, for the impedance of a circuit: Blaquièrè (1960, p. 104).
6 Blaquièrè (1960, II, pp. 102–124). When a signal is linearly transmitted, let us call gain the ratio between the amplitudes of the input and output of Fourier’s terms having the same impulse. For imperfect filters the gain is a function of the impulse. For a (theoretical) filter that should let only three impulses pass, if the operator is the second derivative with respect to time, it applies to the input function

\[ f_i(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \]
to give the output function \( \frac{d^2 f_i(t)}{dt^2} = f_0(t) \): corresponding to the general term of rank \( n \):

\[
t_n = a_n \cos n\omega t + b_n \sin n\omega t
\]

we have:

\[
\frac{d^2}{dt^2} t_n = -n^2 \omega^2 a_n \cos n\omega t - n^2 \omega^2 b_n \sin n\omega t = -n^2 \omega^2 t_0.
\]

With \( n = 3 \)

\[
(2 \times 3) + 1 \text{ terms :}
\]

\[
f_0(t) = 0 \times a_0 - \omega^2(a_1 \cos \omega t - b_1 \sin \omega t - 4\omega^2(a_2 \cos 2\omega t \\
+ b_2 \sin 2\omega t) - 9\omega^2(a_3 \cos 3\omega t + b_3 \sin 3\omega t)
\]


COMMENTS BY PATRICK SUPPES

For many years I have benefitted from conversations with Jules Vuillemin about the history of philosophy and the history of science. The points about which he has instructed and corrected me are many and large in number. He is equally at home in discussing the physics of the Stoics or Aristotle, on the one hand, and on the other hand, the great developments of mathematical physics from the seventeenth to the nineteenth centuries. The present article lays out in some considerable detail the way in which the basic conceptual and mathematical apparatus of quantum mechanics was first developed in the classical theory of optics and electromagnetic fields, these developments in themselves depending upon the theory of waves developed by Huygens and others. Without knowledge of this earlier history of classical physics it is too easy to think that much of classical quantum mechanics is more original in formulation than it is.
JULES VUILLEMIN

However, the really important point brought out by Jules is that in major respects it is not classical mechanics but classical electromagnetic theory, and more generally the theory of wave phenomena, that is the real intellectual predecessor of quantum mechanics. This applies to the theory of operators, as well as the fact that if two operators in quantum mechanics, such as position and momentum, are Fourier transforms of each other, then an inequality of the form of Heisenberg’s uncertainty principle must hold, but initially just in the domain of the classical theory of wave phenomena, especially electromagnetic waves.

A point that Vuillemin does not emphasize, but I think he would agree with, is that quantum mechanics resembles electromagnetic theory more than classical mechanics in other respects. The concept of a particle’s unique trajectory is abandoned. It is well known that it is not possible to compute in classical quantum mechanics even the simple autocorrelation of the positions of a particle at two different times. As I have argued recently in most explicit form, but also earlier as well, quantum mechanics offers from a causal standpoint only a weak theory of the mean probability distributions of particles, and nothing like a full probabilistic theory of sample paths (Suppes, 1990). In classical electromagnetic theory, as Jules points out, there is no concept of a distinct particle path, but only of fields that extend continuously over all of space, in the general case, and over a limited but continuous domain in more restricted cases. It is of great importance, however, to insist on the point that physicists do not really think about electrons, protons, and other particles just in terms of these mean distributions or in terms of the apparatus given to them by classical quantum mechanics. It is completely natural and indeed necessary for the serious detailed discussion of experiments and of almost every kind of microscopic physical phenomenon to think of electrons, protons, and other particles as moving in space with trajectories. The concepts behind the exotic apparatus of high-energy physics are testimony to this. The purpose of linear accelerators is just that, to accelerate particles to very high speeds but it would not make any sense to talk of such acceleration without the particles having trajectories.

There are two remarks I want to make about Vuillemin’s analysis of classical field magnitudes. The first is that the one thing that quantum mechanics did add to the classical theory is the probabilistic interpretation of wave phenomena. This means that what was classically interpreted as wave distributions, became under this interpretation
probability distributions of particles. As far as I know this probabilistic interpretation does not occur anywhere in nineteenth-century physics and is something genuinely new as part of quantum mechanics. What is ironical is that it is the interpretation, not the mathematical formalism, that is the real contribution of quantum mechanics, even though the probabilistic theory of quantum mechanics is a very weak one, in the sense as already mentioned of being only a theory of mean distributions.

The second remark is a historical one. Jules remarks that Kant's treatise *The Metaphysical Foundations of Natural Science* has been unduly neglected by most scholars, except by a few persons like myself. But this is far too modest on his part. I consider myself an amateur scholar of this treatise compared to Vuillemin. He has fortunately caught several errors of mine before they have appeared in print; above all I mention that he himself published the most detailed work (1955) I know of on the *Metaphysical Foundations*.

**REFERENCES**
