Deriving scientific or geometric laws from thought experiments, via meaningfulness, with an application to the Pythagorean Theorem

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What do I mean by a thought experiment?

An abstract constraint represented by a property or an equation, whose validity can be assessed without experimentation

An Example

A (scientific) function

$$G : J \times J' \longrightarrow J : (y, r) \longmapsto G(y, r)$$

with J and J' real intervals.

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A (scientific) function

$$G : J \times J' \longrightarrow J : (y, r) \longmapsto \overbrace{G(\underset{\substack{\uparrow \\ \textit{current} \\ \textit{state}}}{y}\;,\;\underset{\substack{\uparrow \\ \textit{affecting} \\ \textit{variable}}}{r})}^{\textit{output}}$$

The variable r affects y, the current state of the system,
The result is $G(y, r)$.

An Example

A (scientific) function

$$G : J \times J' \longrightarrow J : (y, r) \longmapsto G(y, r)$$

The domain of variation of the variable y is the same as the range of G. So, the function G can be iterated: we can compute $G(G(y, r), t)$.

An Example: Permutability

Our though experiment tells us that the order of the effects does not matter, so:

$$G(G(y,r),t) = G(G(y,t),r).$$

This is called the *permutability condition*.

An Example: Quasi-permutability

Another equation to keep in mind:

$$M(G(y, r), t) = M(G(y, t), r).$$

The function M is *quasi-permutable*.

A consequence of permutability

Under fairly general conditions making empirical sense, the permutability condition implies the existence of a representation

$$G(y, r) = f^{-1}(f(y) + g(r)),$$

where f and g are real valued, strictly monotonic continuous functions. Our result is a generalization an old result of Hosszú (three articles in 1962, see Aczél, 1966).

Permutability + meaningfulness

The equation

$$G(y, r) = f^{-1}(f(y) + g(r)),$$

does not have the form of a scientific law. But if we inject meaningfulness, we can get one.

Meaningfulness is a formal statement ensuring that the form of a law is not affected by a change in the (ratio scales) units.

Many scientific or geometric functions are permutable or quasi permutable. Some examples.

- The Lorentz-FitzGerald Contraction.

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2},$$

$$L(L(p, v), w) = p\left(1 - \left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}}\left(1 - \left(\frac{w}{c}\right)^2\right)^{-\frac{1}{2}} = L(L(p, w), v).$$

Many scientific or geometric functions are permutable or quasi permutable. Some examples.

- BEER'S LAW.

$$I(x,y) = x\,e^{-\frac{y}{c}}.$$

- THE MONOMIAL LAWS, IN SCIENCE AND GEOMETRY.

For example, the volume $C(\ell, r)$ of a cylinder of radius r and height ℓ:

$$C(\ell, r) = \ell \pi r^2.$$

This is permutable. We have:

$$C(C(\ell, r), v) = C(\ell\pi r^2, v) = \ell\pi r^2 \pi v^2 = C(C(\ell, v), r).$$

Other monomial laws are quasi-permutable.

Many scientific or geometric functions are permutable or quasi permutable. Some examples.

- THE PYTHAGOREAN THEOREM.

$$P(x, y) = \sqrt{x^2 + y^2},$$

With $P(x, y)$ the length of the hypothenuse.

This is also permutable. We have:

$$P(P(x, y), z) = \sqrt{P(x, y)^2 + z^2} = \sqrt{x^2 + y^2 + z^2} = P(P(x, z), y).$$

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Here, we get permutability by computation, but you can also get it by a geometric "thought" argument. We will get to that later on.

Many scientific or geometric functions are permutable or quasi permutable. Some examples.

- The Pythagorean Theorem.

$$P(x, y) = \sqrt{x^2 + y^2},$$

But the function P of the Pythagorean Theorem satisfies another condition that can be established by a thought experiment: the function P is symmetric. We have:

$$P(x, y) = P(y, x) \qquad \text{for all positive real numbers } x \text{ and } y.$$

Some basic definitions: Solvability

J, J', and H real nonnegative intervals of positive length.

A *(numerical) code* is a function $M : J \times J' \xrightarrow{\text{onto}} H : (x, t) \mapsto M(x, t)$ strictly increasing x, strictly monotonic in t, and continuous in both.

A code M is *solvable* if it satisfies the two conditions:

S1. If $M(x, t) < p \in H$, there exists $w \in J$ such that $M(w, t) = p$.

S2. The function M is *1-point right solvable*, that is, there exists a point $x_0 \in J$ such that for every $p \in H$, there is $v \in J'$ satisfying $M(x_0, v) = p$. In such a case, we may say that M is *x_0-solvable*.

Some basic definitions: Permutability and Quasi-permutability

A function $M : J \times J' \longrightarrow H$ is *quasi permutable* if there exists a function $G : J \times J' \to J$ comonotonic with M such that

$$M(G(x, s), t) = M(G(x, t), s).$$

We say then that M is *permutable with respect to* G, or *G-permutable*. When M is permutable with respect to itself, we simply say that M is *permutable* (cf. Aczél, 1966).

Theorem

(i) *A solvable code* $M : J \times J' \to H$ *is quasi permutable if and only if there exists three continuous functions* m, f, *and* g, *with* m *and* f *strictly increasing and* g *strictly monotonic, such that*

$$M(y, r) = m(f(y) + g(r)).$$

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(ii) *A solvable code* $G : J \times J' \to J$ *is a permutable code if and only if, with* f *and* g *as above, we have*

$$G(y, r) = f^{-1}(f(y) + g(r)).$$

Theorem

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$$M(y, r) = m(f(y) + g(r)).$$

(ii) *A solvable code $G : J \times J' \to J$ is a permutable code if and only if, with f and g as above, we have*

$$G(y, r) = f^{-1}(f(y) + g(r)).$$

(iii) *If a solvable code $G : J \times J \to J$ is a symmetric function—that is, $G(x, y) = G(y, x)$ for all $x, y \in J$— then G is permutable if and only if there exists a strictly increasing and continuous function $f : J \to J$ satisfying*

$$G(x, y) = f^{-1}(f(x) + f(y)).$$

Sketch of Proof of (ii)

Define the operation $\bullet : J \times J \to J$

$$y \bullet x = G(y, r) \quad \Longleftrightarrow \quad G(x_0, r) = x,$$

Using a generalization of Holder's Theorem (Falmagne, 1975), we get

$$f(y \bullet x) = f(y) + f(x).$$

Defining the strictly monotonic function $\psi : J' \to J$ by

$$\psi(s) = G(x_0, s),$$

we get

$$f(y \bullet x) = f(G(y, r)) = f(y \bullet G(x_0, r)) = f(y) + f(\psi(r)),$$

and so

$$G(y, r) = f^{-1}(f(y) + f(\psi(r))),$$

or with with $g = f \circ \psi$,

$$G(y, r) = f^{-1}(f(y) + g(r)).$$

To sum up

Permutability, quasi permutability, and solvability give us the representations

$$M(y, r) = m(f(y) + g(r))$$

and

$$G(y, r) = f^{-1}(f(y) + g(r)).$$

To go further than that, we must find a way to state, formally, that the form of the laws is invariant under changes in the units.

The ambiguity of the current notation

The standard notation of scientific or geometric laws is ambiguous.

When we represent the Lorentz-FitzGerald Contraction by the equation

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

the units of ℓ and v are not specified: $L(70, 3)$ has no empirical meaning if one does not specify, for example, that the pair $(70, 3)$ refers to 70 meters and 3 kilometers per second, respectively.

The ambiguity of the current notation

$L(70, 3)$ has no empirical meaning if one does not specify, for example, that the pair $(70, 3)$ refers to 70 meters and 3 kilometers per second, respectively.

Such a parenthetical reference to the units is standard in a scientific context, but is not instrumental for our purpose, which is to express, formally, an invariance with respect to any change in the units[1].

[1] A relevant point is made by Pat Suppes on page 120 of *Representation and Invariance of Scientific Structures* (see "Why the Fundamental Equations of Physical Theories Are not Invariant").

How to resolve the ambiguity

Based on an idea of Falmagne and Narens, 1983

Interpret

$$L(\ell, v) \qquad \text{as a shorthand notation for} \qquad L_{1,1}(\ell, v),$$

in which ℓ and L on the one hand, and v on the other hand, are measured in terms of two particular initial or anchor units fixed arbitrarily. Such units could be m (meter) and km/sec, for example.

How to resolve the ambiguity

Based on an idea of Falmagne and Narens, 1983

Changing the units creates a new function

$$L_{\alpha,\beta} \quad \text{which carries the same information as} \quad L = L_{1,1}.$$

Specifically,

$$L_{\alpha,\beta}(\alpha\ell, \beta v) \quad \text{describes the same empirical situation as} \quad L_{1,1}(\ell, v).$$

How to resolve the ambiguity

Based on an idea of Falmagne and Narens, 1983

The connection between L and $L_{\alpha,\beta}$ is thus

$$
\begin{aligned}
\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) &= \left(\frac{1}{\alpha}\right) \alpha\ell \sqrt{1 - \left(\frac{\beta v}{\beta c}\right)^2} \\
&= \ell \sqrt{1 - \left(\frac{v}{c}\right)^2} \\
&= L_{1,1}(\ell, v).
\end{aligned}
$$

This implies, for any α, β, ν and μ in $]0, \infty[$,

$$
\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu} L_{\nu,\mu}(\nu\ell, \mu v).
$$

Needless to say,

I am not suggesting to change the standard notation of scientific laws. However, we can use the extended notation whenever we want to get some juice out of the invariance, as I am doing here.

Defining a *meaningful* collection of codes

$[a, a'[$, $[b, b'[$ and $]d, d'[$ three real non negative intervals.

$$\mathcal{M} = \{M_{\alpha,\beta,\gamma} \,|\, \alpha, \beta, \gamma \in]0, \infty[\,\}$$

a collection of strictly monotonic functions, continuous in both variables

$$M_{\alpha,\beta,\gamma} : [\alpha a, \alpha a'[\, \times [\beta b, \beta b'[\, \xrightarrow{\text{onto}} [\gamma d, \gamma d'[\,.$$

The collection $\mathcal{M}$ is *meaningful* if for all choices of $\alpha, \beta, \gamma, \mu, \nu$ and η in $]0, \infty[$, we have

$$\frac{1}{\gamma} M_{\alpha,\beta,\gamma}(\alpha x, \beta r) = \frac{1}{\eta} M_{\mu,\nu,\eta}(\mu x, \nu r), \qquad (x \in [a, a'[\,;\ r \in [b, b'[\,).$$

The scale of M need not be the same as the scale of x.

Defining a *meaningful* collection of codes

The intervals $[a, a'[$, $[b, b'[$ and $]d, d'[$ may be bounded or unbounded above.

The family $\mathcal{M}$ is *self-transforming* if $[a, a'[= [d, d'[$ and $\alpha = \gamma$ for any function $M_{\alpha,\beta,\gamma}$ in $\mathcal{M}$. Thus, the first input variable and the output variable are measured with the same unit. We write then $M_{\alpha,\beta} = M_{\alpha,\beta,\alpha}$.

The meaningfulness equation becomes

$$\frac{1}{\alpha} M_{\alpha,\beta}(\alpha x, \beta r) = \frac{1}{\mu} M_{\mu,\nu}(\mu x, \nu r), \qquad (x \in [a, a'[\,;\ r \in [b, b'[\,).$$

Theorem (Main result)

Suppose that $\mathcal{G} = \{G_\nu\}$ *is a meaningful self-transforming collection of codes, with*

$$G_\nu :]0,\infty[\times]0,\infty[\xrightarrow{onto}]0,\infty[\, .$$

Suppose also that one of these codes is solvable and permutable with respect to the initial code G*. Moreover, that particular code is symmetric, strictly increasing in both variables, and differentiable with continuous non vanishing derivatives. Then the initial code* G *has necessarily the form:*

$$G(y,x) = \left(y^\theta + x^\theta\right)^{\frac{1}{\theta}} \qquad (y,x \in]0,\infty[)$$

for some $\theta \in]0,\infty[$.

Sketch of proof

$$
\begin{aligned}
&G_\nu(G(\nu y, \nu x), \nu z) \\
&= G_\nu(f^{-1}(f(\nu y) + f(\nu x)), \nu z) && \text{(by previous theorem)} \\
&= \nu G\left(\frac{1}{\nu} f^{-1}(f(\nu y) + f(\nu x)), z\right) && \text{(by meaningfulness)} \\
&= \nu f^{-1}\left(f\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu x)\right)\right) + f(z)\right) && \text{(by previous theorem)} \\
&= \nu f^{-1}\left(f\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu z)\right)\right) + f(x)\right) && \text{(by quasi permutability).}
\end{aligned}
$$

Sketch of proof

$$
\begin{aligned}
&G_\nu(G(\nu y,\nu x),\nu z)\\
&\quad = G_\nu(f^{-1}(f(\nu y)+f(\nu x)),\nu z) && \text{(by previous theorem)}\\
&\quad = \nu G\left(\frac{1}{\nu}f^{-1}(f(\nu y)+f(\nu x)), z\right) && \text{(by meaningfulness)}\\
&\quad = \nu f^{-1}\left(f\left(\frac{1}{\nu}f^{-1}\left(f(\nu y)+f(\nu x)\right)\right)+f(z)\right) && \text{(by previous theorem)}\\
&\quad = \nu f^{-1}\left(f\left(\frac{1}{\nu}f^{-1}\left(f(\nu y)+f(\nu z)\right)\right)+f(x)\right) && \text{(by quasi permutability).}
\end{aligned}
$$

Equating the last two r.h.s's above and simplifying gives

$$
f\left(\frac{1}{\nu}f^{-1}\left(f(\nu y)+f(\nu x)\right)\right)+f(z) = f\left(\frac{1}{\nu}f^{-1}\left(f(\nu y)+f(\nu z)\right)\right)+f(x).
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Equating the last two r.h.s's above and simplifying gives

$$f\left(\frac{1}{\nu}f^{-1}\left(f(\nu y)+f(\nu x)\right)\right)+f(z)=f\left(\frac{1}{\nu}f^{-1}\left(f(\nu y)+f(\nu z)\right)\right)+f(x).$$

Differentiating with respect to x and z leads to the Pexider equation

$$f'(\nu x)=f'(x)h(\nu),$$

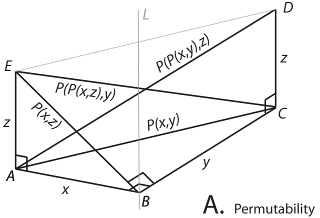
with solution $f'(x)=\kappa x^{\zeta}$ and $h(\nu)=\nu^{\zeta}$. We obtain with $\xi=\frac{\kappa}{\theta}$ and $\theta=\zeta-1$, $f(x)=\xi x^{\theta}$, and thus

$$G(y,x)=f^{-1}(f(y)+f(x))=\left(\frac{\xi y^{\theta}+\xi x^{\theta}}{\xi}\right)^{\frac{1}{\theta}}=\left(y^{\theta}+x^{\theta}\right)^{\frac{1}{\theta}}.$$

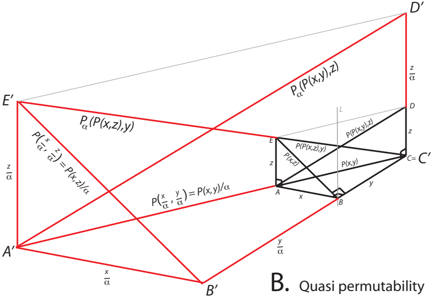
□

The thought experiment is a simple geometrical argument.

In Figure A, the two hypothenuses of length $P(P(x,y),z)$ and $P(P(x,z),y)$ are coplanar.
They are the two diagonals of the rectangle $ACDE$.
In Figure B, the quasi permutability creates a similar figure.



A. Permutability



B. Quasi permutability

Where the idea came from.

By the way, the idea of applying these concepts to geometry was suggested to me by — who else? — Pat Suppes.

The End: Thank you Pat, and

Happy Birthday!