The Incompleteness Of Hölder’s Theorem

During Most of The 20th Century

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Hölder’s (1901) Theorem

- One very important measurement result involved structures $\langle X, \preceq, \odot \rangle$ where $X$ is a set of, e.g., homogeneous masses, $\preceq$ is an ordering of masses using a pan balance, and $\odot$ is a binary concatenation of masses, placing two masses on a pan of the balance.

- Hölder’s (1901) theorem, formalizing some of Helmholtz’s earlier ideas, stated axioms about $\langle X, \preceq, \odot \rangle$ such that when mapped via an order preserving function $\varphi$ into $\langle \mathbb{R}^+, \geq, + \rangle$, then

\[ \varphi(x \odot y) = \varphi(x) + \varphi(y). \]
Hölder’s (1901) Theorem Continued

- He also considered the multiplicative conjoint representation of mass as volume times density.

- And he assumed distribution conditions that forced a common measure \( \varphi \) of mass for both concatenation and conjoint scales.

- But, given that this involved both addition and multiplication, why did he map \( \langle X, \preceq, \odot \rangle \) just into \( \langle \mathbb{R}^+, \geq, + \rangle \) rather than into the full real numbers \( \langle \mathbb{R}^+, \geq, +, \times \rangle \)?
What He and the Rest of Us Missed for Most of 20\textsuperscript{th} Century

- In fact, if we do map into $\langle \mathbb{R}^+, \geq, +, \times \rangle$, then his axioms about $\langle X, \succsim, \odot \rangle$ lead to a representation $\varphi$ such that

$$
\varphi(x \odot y) = \varphi(x) + \varphi(y) + \delta \varphi(x) \varphi(y), \quad \delta = -1, 0, 1.
$$

(Luce, 1991, 2000).

- This representation is called \textit{polynomial-additive} (for short p-additive) because it is the only polynomial that can be transformed into an additive representation (Aczél, 1966).
p-Additivity

- Clearly, for $\delta = 0$, p-additivity is, in fact, additive.

- The distribution laws of the physical applications force $\delta = 0$. So the cases $\delta = -1$ and $1$ do not matter for physics.

- But they very much do matter for the behavioral sciences, as I outline next.
Testable Predictions From Luce’s (2004) Global Psychophysical Theory: Primitives for Binary Receptors

- The ears, eyes, and arms are binary receptors that work as teams.

- Let $X$ denote the set of physical intensities less their respective threshold intensities. (Not dB!) If $x, u \in X$, then the stimulus presented is $(x, u) \in X \times X$.

- The order $\preceq$ over $X \times X$ by subjective intensity is assumed to be a weak order
Binary Primitives Continued

- Suppose that a respondent matches \((z, z)\) to \((x, u)\), then define the operator \(\oplus\) by

  \[
x \oplus u := z.
  \]

- The nature of \(\oplus\) characterizes the trade-off between the binary receptors.
• Suppose also that the experimenter presents signal $x$ and a number $p$, and assume there is a reference signal $\rho < x$.

• The respondent reports the signal $y$ such that the “interval from from the reference signal $\rho$ to $y$” is perceived as $p$ “times” as intense as the “interval from $\rho$ to $x$”.

  – It is convenient to think of $y$ as an operator: $y = x \circ_p \rho$.

  – When $\rho = 0$, this is nothing but S. S. Stevens’ method of magnitude production.
Linking $\oplus$ and $\circ_p$

- Luce (2004, 2008) formulated behavioral (i.e., testable) axioms among the primitives.

- Included were two linking properties between matching and production somewhat analogous to the distribution properties in physics.

- These allow us to have the same psychophysical function for matching and production.
Representations of Binary Intensities

- These assumed properties imply the following representation:

  - A \textit{p–additive} order preserving psychophysical function \( \psi \):
    \[
    \psi(x \oplus y) = \psi(x \oplus 0) + \psi(0 \oplus y) + \delta \psi(x \oplus 0) \psi(0 \oplus y), \quad \delta = -1, 0, 1,
    \]
  - with
    \[
    \psi(x \oplus 0) = \gamma \psi(0 \oplus x).
    \]
  - And a weighting function \( W \) over positive numbers such that
    \[
    W(p) = \frac{\psi(x \circ_p \rho) - \psi(\rho)}{\psi(x) - \psi(\rho)}.
    \]
Representations of Binary Intensities Continued

- For loudness and using individual respondents, Steingrimsson & Luce (2005a, 2005b, 2006, 2007) strongly supported the behavioral axioms

- Steingrimsson (2009, 2011, in preparation) has and is running a parallel series for brightness and for “perceived” contrast
  - Equally strongly supported.
Form of the Psychophysical Function $\psi$

- A simple behavioral invariance implies $\psi$ is a power function
  \[ \psi(x) = \alpha x^\beta. \]

- The invariance has been empirically supported for loudness and brightness in the third article of above sequences.

- In the 1960s and 1970s this was thought to be sustained for all intensity (prothetic) attributes (Stevens and others).
  
  - Using geometric averaging, the log-log plots of binary attributes were “plausibly” linear. Caution: fitted lines do guide one’s eyes.
Several Familiar Operator Properties

- **Commutativity:** \( x \oplus u \sim u \oplus x \).
  - Rejected (loudness, brightness, perceived contrast) for individuals.

- **Associativity:** \((x \oplus u) \oplus v \sim x \oplus (u \oplus v)\).
  - No data so far. Shortly we’ll see why we expect it to fail.

- **Bisymmetry:** \((x \oplus y) \oplus (u \oplus v) \sim (x \oplus u) \oplus (y \oplus v)\).
  - Accepted (loudness, brightness, contrast) for individuals. Not yet tested for two-arm weight lifting.
Predictions of the Binary Theory

• Suppose the p-additive representation holds. Then I showed (Psychological Review, in press):

  – For $\delta = 0$, bisymmetry is satisfied.

  – For $\delta \neq 0$, bisymmetry is satisfied iff commutativity is satisfied. Conclusion: The data – Yes to bisymmetry, No to commutativity – imply $\delta = 0$, i.e. simple additivity.

  – This corrects Luce’s (2004) unconditional claim of $\delta = 0$. The error was pointed out by C. T. Ng (Luce 2008).

  – If bisymmetry holds in this context, associativity cannot hold.
Unary Theory

• Many intensity senses are unary, not binary. Examples: taste, electric shock, vibration, force, linear extent, preference for money, etc.

• Consider those cases where a signal concatenation ⊙ exists and has an additive physical ratio scale representation.
  
  – So ⊙ satisfies both commutativity and associativity.
Unary Magnitude Production

• Exactly as with the binary theory, we assume magnitude production and a linking axiom between the $\circ_p$ production structure and the $\odot$ structure.

  – And that means we definitely need $\langle \mathbb{R}^+, \geq, +, \times \rangle$, so we have to allow the $p$-additive form.

• And, quite unlike the binary case, $\odot$ satisfies all of: bisymmetry, commutativity, and associativity.

  – I do not know of any principled argument that forces the $\delta = 0$ case.
p-Additive Scale Types

- For the additive case ($\delta = 0$), $\varphi$ is a ratio scale.

- For the non-additive cases ($\delta = -1, 1$), p-additivity is equivalent to

  \[ 1 + \delta \varphi(x \odot y) = [1 + \delta \varphi(x)] [1 + \delta \varphi(y)] \]

  - Clearly, $\ln$ of this is additive.

  - Because $\delta = -1, 1$, $\varphi$ must be an absolute, not a ratio, scale.
Form of the Psychophysical Function $\varphi$

- Unlike the binary case, there are 3 types corresponding to the value of $\delta$, and Luce (2012, in press) shows

\[
\begin{align*}
\varphi_0(x) &= \eta x \quad (\eta > 0) \quad \text{if } \delta = 0 \\
\varphi_1(x) &= e^{\lambda x} - 1 \quad (\lambda > 0) \quad \text{if } \delta = 1 \\
\varphi_{-1}(x) &= 1 - e^{-\kappa x} \quad (\kappa > 0) \quad \text{if } \delta = -1
\end{align*}
\]

- For $\delta = 0$, this is a special case of a power function.

- For $\delta \neq 0$, clearly not power functions.
But the Claim is Just Power Functions

• The empirical literature seemed to defend power functions; what gives?

• For averaged cross-modal matches between loudness of noise, vibration, and shock each measured in dB:
  
  – loudness versus vibration seemed to be a power function – but fitted lines can deceive.

  – shock versus loudness and versus vibration were equally not power functions, although Stevens tried – not very convincingly – to explain that fact away.
Predictions of Cross-Modal Matches

- From the binary and unary theories, it is fairly apparent that predictions of cross-modal mappings should follow

  - They do.

  - They are moderately complicated because of the unary case’s 3 representations $\delta = -1, 0, 1$.

  - Because we know of no $\delta = -1$ attribute other than utility of money for some people, that case is omitted in the following already complex table.
## Cross-Modal Matches

Match $\varphi_b(z)$ to $\varphi_a(x)$

<table>
<thead>
<tr>
<th></th>
<th>binary</th>
<th>unary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_b = 0$</td>
<td>$\delta_b = 0$</td>
<td>$\delta_b = 1$</td>
</tr>
</tbody>
</table>

- **binary**
  - $\delta_a = 0$
  - $x$
  - power

- **unary**
  - $\delta_a = 1$
  - $\frac{1}{\alpha_b} \left( e^{\lambda_a x} - 1 \right)^{\frac{1}{\beta_b}}$
  - proportion

- **unary**
  - $\delta_a = -1$
  - $\frac{1}{\alpha_b} \left( 1 - e^{-\kappa_a x} \right)^{\frac{1}{\beta_b}}$
  - proportion

\[
\begin{align*}
\frac{1}{\lambda_b} \ln \left( 1 + \eta_a x \right)^{\frac{1}{\beta_b}}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\eta_b} \left( e^{\lambda_a x} - 1 \right) & \quad \text{proportion} \\
\frac{1}{\eta_b} \left( 1 - e^{-\kappa_a x} \right) & \quad \text{proportion}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\lambda_b} \ln \left( 2 - e^{-\kappa_a x} \right)
\end{align*}
\]
Observations

• The unary predictions differ from a power function only when $\delta = \pm 1$.

• I suspect that further research will confirm that for shock, pain, and vibration $\delta \neq 0$.

• Utility of money, which is unusual because the domain includes losses and well as gains, appears to be a case where all 3 can occur: risk seeking, risk neutral, and risk averse types.
Closing Remarks

• The overlooked solutions to Hölder’s axiomatizations did not matter for physics, but they certainly appear to matter greatly for the behavioral and economic sciences.

• Much experimentation is needed to check these predictions.

• Also experiments must be analyzed for individual respondents; averaging respondents is clearly inappropriate.
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