The Incompleteness Of Hölder's Theorem

During Most of The 20th Century

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Hölder's (1901) Theorem

- One very important measurement result involved structures $\langle X, \succsim, \odot \rangle$ where X is a set of, e.g., homogeneous masses, $\succsim$ is an ordering of masses using a pan balance, and $\odot$ is a binary concatenation of masses, placing two masses on a pan of the balance.

- Hölder's (1901) theorem, formalizing some of Helmholtz's earlier ideas, stated axioms about $\langle X, \succsim, \odot \rangle$ such that when mapped via an order preserving function φ into $\langle \mathbb{R}^+, \geq, + \rangle$, then

$$\varphi(x \odot y) = \varphi(x) + \varphi(y).$$

Hölder's (1901) Theorem Continued

- He also considered the multiplicative conjoint representation of mass as volume times density.

- And he assumed distribution conditions that forced a common measure φ of mass for both concatenation and conjoint scales.

- But, given that this involved both addition and multiplication, why did he map $\langle X, \succsim, \odot \rangle$ just into $\langle \mathbb{R}^+, \geq, + \rangle$ rather than into the full real numbers $\langle \mathbb{R}^+, \geq, +, \times \rangle$?

What He and the Rest of Us Missed for Most of 20^{th} Century

- In fact, if we do map into $\langle \mathbb{R}^+, \geq, +, \times \rangle$, then his axioms about $\langle X, \succsim, \odot \rangle$ lead to a representation φ such that

$$\varphi(x \odot y) = \varphi(x) + \varphi(y) + \delta\varphi(x)\varphi(y), \qquad \delta = -1, 0, 1.$$

(Luce, 1991, 2000).

- This representation is called *polynomial-additive* (for short p-additive) because it is the only polynomial that can be transformed into an additive representation (Aczél, 1966).

p-Additivity

- Clearly, for $\delta = 0$, p-additivity is, in fact, additive.

- The distribution laws of the physical applications force $\delta = 0$. So the cases $\delta = -1$ and 1 do not matter for physics.

- But they very much do matter for the behavioral sciences, as I outline next.

Testable Predictions From Luce's (2004) Global Psychophysical Theory: Primitives for Binary Receptors

- The ears, eyes, and arms are binary receptors that work as teams.

- Let X denote the set of physical intensities less their respective threshold intensities. (Not dB!) If $x, u \in X$, then the stimulus presented is $(x, u) \in X \times X$.

- The order $\succsim$ over $X \times X$ by subjective intensity is assumed to be a weak order

Binary Primitives Continued

- Suppose that a respondent matches (z, z) to (x, u), then define the operator $\oplus$ by

$$x \oplus u := z.$$

- The nature of $\oplus$ characterizes the trade-off between the binary receptors.

Binary Primitives Continued

- Suppose also that the experimenter presents signal x and a number p, and assume there is a reference signal $\rho < x$.

- The respondent reports the signal y such that the "interval from from the reference signal ρ to y" is perceived as p "times" as intense as the "interval from ρ to x".
  - It is convenient to think of y as an operator: $y = x \circ_p \rho$.
  - When $\rho = 0$, this is nothing but S. S. Stevens' method of magnitude production.

Linking $\oplus$ and $\circ_p$

- Luce (2004, 2008) formulated behavioral (i.e., testable) axioms among the primitives.

- Included were two linking properties between matching and production somewhat analogous to the distribution properties in physics.

- These allow us to have the same psychophysical function for matching and production.

Representations of Binary Intensities

- These assumed properties imply the following representation:
  - A *p–additive* order preserving psychophysical function ψ:

$$\psi(x \oplus y) = \psi(x \oplus 0) + \psi(0 \oplus y) + \delta\psi(x \oplus 0)\psi(0 \oplus y), \quad \delta = -1, 0, 1,$$

  - with

$$\psi(x \oplus 0) = \gamma\psi(0 \oplus x).$$

  - And a weighting function W over positive numbers such that

$$W(p) = \frac{\psi(x \circ_p \rho) - \psi(\rho)}{\psi(x) - \psi(\rho)}.$$

Representations of Binary Intensities Continued

- For loudness and using individual respondents, Steingrimsson & Luce (2005a, 2005b, 2006, 2007) strongly supported the behavioral axioms

- Steingrimsson (2009, 2011, in preparation) has and is running a parallel series for brightness and for "perceived" contrast
  - Equally strongly supported.

Form of the Psychophysical Function ψ

- A simple behavioral invariance implies ψ is a power function

$$\psi(x) = \alpha x^{\beta}.$$

- The invariance has been empirically supported for loudness and brightness in the third article of above sequences.

- In the 1960s and 1970s this was thought to be sustained for all intensity (prothetic) attributes (Stevens and others).

  – Using geometric averaging, the log-log plots of binary attributes were "plausibly" linear. Caution: fitted lines do guide one's eyes.

Several Familiar Operator Properties

- **Commutativity:** $x \oplus u \sim u \oplus x$.
  - Rejected (loudness, brightness, perceived contrast) for individuals.

- **Associativity:** $(x \oplus u) \oplus v \sim x \oplus (u \oplus v)$.
  - No data so far. Shortly we'll see why we expect it to fail.

- **Bisymmetry:** $(x \oplus y) \oplus (u \oplus v) \sim (x \oplus u) \oplus (y \oplus v)$.
  - Accepted (loudness, brightness, contrast) for individuals. Not yet tested for two-arm weight lifting.

Predictions of the Binary Theory

- Suppose the p-additive representation holds. Then I showed (*Psychological Review*, in press):
  - For $\delta = 0$, bisymmetry is satisfied.
  - For $\delta \neq 0$, bisymmetry is satisfied iff commutativity is satisfied. Conclusion: The data – Yes to bisymmetry, No to commutativity – imply $\delta = 0$, i.e. simple additivity.
  - This corrects Luce's (2004) unconditional claim of $\delta = 0$. The error was pointed out by C. T. Ng (Luce 2008).
  - If bisymmetry holds in this context, associativity cannot hold.

Unary Theory

- Many intensity senses are unary, not binary. Examples: taste, electric shock, vibration, force, linear extent, preference for money, etc.

- Consider those cases where a signal concatenation $\odot$ exists and has an additive physical ratio scale representation.
  - So $\odot$ satisfies both commutativity and associativity.

Unary Magnitude Production

- Exactly as with the binary theory, we assume magnitude production and a linking axiom between the $\circ_p$ production structure and the $\odot$ structure.
  - And that means we definitely need $\langle \mathbb{R}^+, \geq, +, \times \rangle$, so we have to allow the p-additive form.

- And, quite unlike the binary case, $\odot$ satisfies all of: bisymmetry, commutativity, and associativity.
  - I do not know of any principled argument that forces the $\delta = 0$ case.

p-Additive Scale Types

- For the additive case ($\delta = 0$), φ is a ratio scale.

- For the non-additive cases ($\delta = -1, 1$), p-additivity is equivalent to

$$1 + \delta\varphi(x \odot y) = [1 + \delta\varphi(x)]\,[1 + \delta\varphi(y)]$$

  - Clearly, ln of this is additive.

  - Because $\delta = -1, 1$, φ must be an **absolute**, not a ratio, scale.

Form of the Psychophysical Function φ

- Unlike the binary case, there are 3 types corresponding to the value of δ, and Luce (2012, in press) shows

$$\begin{array}{ll} \varphi_0(x) = \eta x \quad (\eta > 0) & \text{if } \delta = 0 \\ \varphi_1(x) = e^{\lambda x} - 1 \;\; (\lambda > 0) & \text{if } \delta = 1 \\ \varphi_{-1}(x) = 1 - e^{-\kappa x} \;\; (\kappa > 0) & \text{if } \delta = -1 \end{array} .$$

  - For $\delta = 0$, this is a special case of a power function.

  - For $\delta \neq 0$, clearly not power functions.

But the Claim is Just Power Functions

- The empirical literature seemed to defend power functions; what gives?

- For averaged cross-modal matches between loudness of noise, vibration, and shock each measured in dB:
  - loudness versus vibration seemed to be a power function – but fitted lines can deceive.
  - shock versus loudness and versus vibration were equally not power functions, although Stevens tried – not very convincingly – to explain that fact away.

Predictions of Cross-Modal Matches

- From the binary and unary theories, it is fairly apparent that predictions of cross-modal mappings should follow
  - They do.
  - They are moderately complicated because of the unary case's 3 representations $\delta = -1, 0, 1$.
  - Because we know of no $\delta = -1$ attribute other than utility of money for some people, that case is omitted in the following already complex table.

Cross-Modal Matches

		Match $\varphi_b(z)$ to $\varphi_a(x)$		
		binary	unary	
		$\delta_b = 0$	$\delta_b = 0$	$\delta_b = 1$
binary	$\delta_a = 0$	power	power	$\left[\frac{1}{\lambda_b} \ln\left(1 + \eta_a x^{\beta_a}\right)\right]^{\frac{1}{\beta_b}}$
x				
	$\delta_a = 0$	power	proportion	$\left[\frac{1}{\lambda_b} \ln\left(1 + \eta_a x\right)\right]^{\frac{1}{\beta_b}}$
unary	$\delta_a = 1$	$\left[\frac{1}{\alpha_b}\left(e^{\lambda_a x} - 1\right)\right]^{\frac{1}{\beta_b}}$	$\frac{1}{\eta_b}\left(e^{\lambda_a x} - 1\right)$	proportion
	$\delta_a = -1$	$\left[\frac{1}{\alpha_b}\left(1 - e^{-\kappa_a x}\right)\right]^{\frac{1}{\beta_b}}$	$\frac{1}{\eta_b}\left(1 - e^{-\kappa_a x}\right)$	$\frac{1}{\lambda_b}\ln\left(2 - e^{-\kappa_a x}\right)$

Observations

- The unary predictions differ from a power function only when $\delta = \pm 1$.
- I suspect that further research will confirm that for shock, pain, and vibration $\delta \neq 0$.
- Utility of money, which is unusual because the domain includes losses and well as gains, appears to be a case where all 3 can occur: risk seeking, risk neutral, and risk averse types.

Closing Remarks

- The overlooked solutions to Hölder's axiomatizations did not matter for physics, but they certainly appear to matter greatly for the behavioral and economic sciences.

- Much experimentation is needed to check these predictions.

- Also experiments must be analyzed for individual respondents; averaging respondents is clearly inappropriate.

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