REPRESENTATIONAL MEASUREMENT IN ECONOMICS*

ADOLFO GARCÍA DE LA SIENRA
Instituto de Filosofía
Facultad de Economía
Universidad Veracruzana
asienrag@gmail.com

My aim in the present paper is to discuss the role of the representational theory of measurement (RTM from now on) in economics. According to Luce and Suppes (2002),

> Representational measurement is, on the one hand, an attempt to understand the nature of empirical observations that can be usefully recoded, in some reasonably unique fashion, in terms of familiar mathematical structures. [. . .] representational measurement goes well beyond the mere construction of numerical representations to a careful examination of how such representations relate to one another in substantive scientific theories, such as in physics, psychophysics, and utility theory. These may be thought of as applications of measurement concepts for representing various kinds of empirical relations among variables.[1]

Suppes' beautiful example of measuring weights by means of a balance (quoted by Morgan 2007, p. 106), as represented by a suitable set-theoretical structure, suggests that RTM involves a claim about measuring procedures (perhaps a form of instrumentalism), and that every measuring procedure consists of constructing some homomorphism. Curiously enough, other authors have got the impression that the set-theoretical structures RTM makes use of, far from representing empirical instrumentalist measurement procedures, are not even about the "natural world". Michell (2007, p. 36), for one, writes that RTM

* The present paper was produced with the support of CONACYT Project 127380, Filosofía de la Economía.

[1] By far, the most important presentation of RTM is found in Krantz, Luce, Suppes and Tversky 2007.

> is based upon an inconsistent triad: first, there is the idea that mathematical structures, including numerical ones, are about abstract entities and not about the natural world; second, there is the idea that representation requires at least a partial identity of structure between the system represented and the system representing it; and third, there is the idea that measurement is the numerical representation of natural systems. The second and third ideas imply that natural systems instantiate mathematical structures and when the natural system involves an unbounded, continuous quantity, it provides an instance of the system of positive real numbers. Thus the second two refute the first idea, the principal *raison d'être* for the representational theory.

Let me start by distinguishing between metrization and measuring. By 'metrization of magnitude *M*' I understand a demonstration, out of empirically meaningful conditions, of the fact that *M* is measurable. RTM accomplishes the metrization of *M* by building suitable set-theoretical structures and proving the existence of a certain homomorphism, which is a mathematical representation of *M*. By 'measuring procedure' I understand a procedure for actually finding the value of *M* for given objects, with respect to a unit of reference. It seems to me that RTM *does not* claim that every measuring procedure consists of explicitly constructing some homomorphism. What it *does* claim is that any (putative) magnitude *M* must satisfy certain conditions in order for it to be metrizable (Díez 2000, p. 20, n. 5). The task of the theory of fundamental metrization is to probe into the conditions that *M* must satisfy in order to guarantee the existence of such representation when the same does not presuppose any other previous metrizations. The fundamental measuring procedures determine specific empirical procedures for qualitative comparison of the specific property involved, and chooses a standard with which the assignment begins (ibid).

It is certain that RTM has never maintained the claim that "mathematical structures, including numerical ones, are about abstract entities and not about the natural world". Clearly, mathematical structures *are* abstract entities, insofar as they are set-theoretical structures. And they *are* "about" the natural (and the "social") world, but the meaning of 'about' in this context needs philosophical clarification. I will try to clarify the idea that natural systems "instantiate" mathematical structures, and criticize the idea that natural systems may "involve" *unbounded*, continuous quantities. It will be seen that the existence of important metrizations in economic theory requires the adoption of rather severe restrictions.

1 Prolegomena

Let me start by making some distinctions. I distinguish target (concrete, real) systems, given in pre-theoretical experience, from model systems on one hand, and set-theoretical structures, on the other. I take for granted that scientific theories can be identified with certain ordered classes of set-theoretical structures (cf. Suppes 2002; Balzer, Moulines and Sneed 1987). Moreover, I assume that theories can be formulated by means of the definition of set-theoretical predicates (cf. Suppes 2002).

The relationships between target systems, model systems and set-theoretical structures is roughly the following: a model system is an idealized replica of a certain class of target systems, a replica that can be described by means of axioms defining a set-theoretical predicate. Model systems are not set-theoretical structures but rather (imagined) physical, economic or geometric objects that represent their corresponding target systems. Thus models systems can be seen

> as imagined physical systems, i.e. as hypothetical entities that, as a matter of fact, do not exist spatio-temporally but are nevertheless not purely mathematical or structural in that they would be physical things if they were real. If the Newtonian model system of sun and earth were real, it would consist of two spherical bodies with mass and other concrete properties such as hardness and colour, properties that structures do not have; likewise, the populations in the Lotka-Volterra model would consist of flesh-and-blood animals if they were real, and the agents in Edgeworth's economic model would be rational human beings. (Frigg 2010, p. 253)

Hence, it is important to stress that the representation of a target system by means of a model system cannot be identified with any kind of homomorphism, since homomorphisms are mappings from one set-theoretical structure into another. My version and defense of RTM is grounded upon these distinctions.

2 Empirical Structures?

When the question arises whether the property of components of a given type of concrete system is measurable, RTM proceeds to build a typical model system to represent that class and, based upon the properties of such model, builds a class of set-theoretical structures by means of the definition of a set-theoretical predicate. It is in this sense that mathematical structures *are* about the natural world: they are true of (more or less) idealized model systems that purport to *represent* some type of concrete target system in the world. These structures, insofar as they represent target systems, may be called "empirical structures".

Suppes' famous example of measuring weights by means of a balance is a classical instance of an "empirical structure", but it can be misleading in two ways. It may suggest both that measurement structures are always empirical, in the sense that they are not describing idealized model systems, and that RTM is a general theory of concrete empirical measuring procedures. The main criticisms against RTM in the field of economic methodology[2] begin by misconstruing it as a theory of measuring procedures, which it is not except for some cases: It is a theory about the conditions that properties of certain classes of target systems must satisfy in order to be measurable. It functions, thus, as a sort of ideal control for empirical measuring procedures (that can also be suggestive of empirical measuring procedures).

A natural or social system (a target system) "instantiates" a mathematical structure if, and only if, it is exactly described by the axioms defining the structure (under a certain empirical interpretation) without the mediation of a model system. In such a case, the structure may be called an "empirical structure". For instance, a structure $\mathfrak{A} = \langle X, \succsim \rangle$ would be an empirical structure faithfully recoding the preferences of a given consumer if X is interpreted as the collection of consumption menus over which he

[2] Cf. Boumans 2007b, pp. 7–8. Boumans clearly thinks that RTM claims something like that it is impossible to measure some empirical magnitude if an axiomatic system of the proper sort is not provided. See also Boumans 2008.

has to make a choice and his pairwise preferences are correctly recorded by $\succsim$. But then nothing guarantees a priori that $\mathfrak{A}$ will be a weak order.

In general, it is not target systems the objects that are exactly described by mathematical structures, but rather model systems representing the former. Hence, many mathematical structures that we find in economics are not instantiated by target systems, but only by model systems obtained out of idealized conditions.

It is usual, in the process of elaboration of economic theories, the construction of ideal objects out of concepts that, even though they originate in experience and correspond to real properties of real beings, have suffered a sort of deformation. For example, there is no doubt that real economic agents are human beings who handle certain information, who can remember events, and that can also perform arithmetic operations. That is why the predicates

'x possesses information' (1)

'x has memory' (2)

and

'x is able to perform calculations' (3)

are true of human beings who effect economic transactions anywhere. Nevertheless, even though economic theory has to refer somehow to the properties they denote, the use of the mathematical method requires the use of deformed versions of such predicates, or of the concepts they express thereby. Some specializations of economic theory require, for instance, for the economic agents to possess perfect information, perfect memory and unlimited computational powers. Thus, for instance, predicates (1), (2) y (3) are deformed in order to obtain

'x possesses perfect information' (4)

'x has perfect recall' (5)

and

'x has unlimited computational powers'. (6)

The conjunction of predicates

$$\text{`}x \text{ posesses perfect information} \wedge x \text{ has perfect recall} \wedge \\ \wedge\, x \text{ has unlimited computational powers'} \quad (7)$$

defines a type or ideal object — what we have called a model system — nonexistent in reality but required to channel mathematical reasoning. As Walras (1954, p. 71) put it:

> the mathematical method is not an *experimental* method; it is a *rational* method, . . . the pure science of economics should then abstract and define ideal-type concepts in terms of which it carries its reasoning. The return to reality should not take place until the science is completed and then only with a view to practical applications.

Let us call 'idealized concepts' (or, briefly, idealizations) the concepts obtained by deformation out of concepts which are truly predicable of real beings if, in spite of this deformation, the same are meaningful — even if they are false — of these real objects, and keep an intension akin to those. We have called 'ideal objects' those model systems built by means of conjunctions of predicates that express idealizations.

The axioms of a consistent economic theory define directly a set-theoretical predicate (as '$\mathfrak{A}$ is a regular preference structure') but they are also obliquely true, at least under a typical interpretation of the primitive terms, of ideal objects. For instance, the just mentioned predicate can be defined canonically as follows: $\mathfrak{A}$ is a *regular preference structure* iff $\mathfrak{A}$ is a weak order. It is debatable whether the preference relation (partially) exhibited by a human agent at a given moment is connected and transitive. Nevertheless, the systematic development of the theory can go on as if it were so.

Utility theory has proceeded through the construction of ideal objects endowed with idealized properties. It has assumed, for instance, that the "consumer" (an ideal object) defines his preference relation over a space of consumption menus which is representable as the nonnegative orthant Ω of $\mathbb{R}^L$. Other idealizations introduced over and above these are those of a continuous or unsatiated preference relation. All these idealizations are needed in order to prove the existence of utility functions possesss-ing certain characteristics required by economic theory (like continuity or monotonicity).

Hence, the use of RTM to obtain such functions cannot be seen as a case of empirical measurement. It is usual in economics the application of RTM methods to obtain representations of idealized magnitudes. For instance, Beaver and Demski (1979) used such methods in trying to fundamentally measure income. They found that income can be fundamentally measured only in a world of complete and perfect markets, but not necessarily otherwise. They concluded that "at a fundamental level the central feature of financial reporting cannot be income measurement".

3 Unbounded Quantities

Since there are no natural systems involving unbounded, continuous quantitites, as this would require the existence of infinitely extended physical systems, no natural target system provides an instance of the system of positive real numbers.

If a system can be represented by means of an unbounded subset of $\mathbb{R}^n$, then this system has to be an idealization. Not even the whole universe is — according to contemporary cosmology — infinitely extended, as Bruno or Newton thought. Indeed, Newton's conception of physical space as absolute describes a system model that corresponds to no concrete target system, since there are no infinitely extended natural systems (in any sense). The same can be said about Newton's model of time: if the universe had a beginning and will have an end, the real line cannot serve to model physical time.

Yet, as a matter of fact, RTM is not committed to deny the Newtonian views of space and time. But it is not committed to assert them, either. If Newtonian space and time do exist, then they would instantiate a set-theoretical structure representing them by means of copies of the real number system. RTM provides an appropriate conceptual framework to formulate, in a precise fashion, this thesis or its opposite.

The problem of the continuum as a property of physical objects was considered by Leibniz as a labyrinth.[3] From the point of view of RTM, any attempt at representing the continuum as a natural entity or quality should proceed by characterizing it by means of plausible intuitive axioms and, once this has been done, by proving that such axioms do imply the desired representation. For instance, someone might wish to employ the conceptual apparatus given by Clark (1985) to describe continuous physical systems or entities. The existence of a representation by means of subsets of $\mathbb{R}^n$ then becomes a precisely formulated problem; but the point is that this is feasible thanks to the versatility of RTM.

4 Models of Data in Measuring Procedures

The view of theories as systems of set-theoretical structures naturally leads to a particular view of theory-application and testing. Typically, to conceptualize a certain phenomenon means to "apply" the conceptual apparatus of the theory fleshing out the terms with a particular concrete meaning. For instance, in order to conceptualize a target system c, a real human consumer, demand theory (T) claims that c be represented by a certain model system m, which is a consumer with unlimited memory that knows beforehand what would be his election in confronting any budget set $\{\mathbf{x} \in X \mid \mathbf{px} \leq w\}$, where $\mathbf{p} \in P$ is a system of prices and $w \in W$ is his income. The information of this ideal consumer is represented in a set-theoretical structure by means of a demand function $\eta{:}\mathcal{B} \to X$. If the Slutsky matrix corresponding to η is symmetric and negative-semidefinite, m knows that η coincides with a Walrasian demand function μ derived from a utility function that represents his preferences; moreover, since m has unlimited computational capabilities, he can determine instantaneously the preference relation from which μ (which turns out to be η) derives.

If the theory represents c with m, the scientist may try to apply T to c. Whether T is applicable or not to c is an entirely empirical matter. To say

[3] "There are two labyrinths of the human mind: one concerns the composition of the continuum, and the other the nature of freedom" (*Leibniz: Philosophical Writings*, p. 107; quoted by Brown 1984, p. 17).

Argument	Value of $\hat{\eta}$
$(\mathbf{p}^1, w^1)$	$\hat{\mathbf{x}}^1 = \hat{\eta}(\mathbf{p}^1, w^1)$
$(\mathbf{p}^2, w^1)$	$\hat{\mathbf{x}}^2 = \hat{\eta}(\mathbf{p}^2, w^2)$
$\vdots$	$\vdots$
$(\mathbf{p}^k, w^k)$	$\hat{\mathbf{x}}^n = \hat{\eta}(\mathbf{p}^k, w^k)$
$\vdots$	$\vdots$
$(\mathbf{p}^N, w^N)$	$\hat{\mathbf{x}}^N = \hat{\eta}(\mathbf{p}^N, w^N)$

that it is is a *claim* that has to be held on the basis of empirical data. Varian (1983) has stressed that there are two approaches to this problem. One, the approach based upon calculus, originated in the work of Antonelli (1986) and Slutski (1915), derives necessary and sufficient conditions on the derivatives of the demand function η. The second, originated in the work of Samuelson (1938, 1947, 1948) is called 'non-parametric' because it assumes no specific form whatsoever for η.

> The distinction between the two approaches is very important in empirical work. The calculus approach assumes the entire demand function [our η] is available for analysis, while the algebraic approach assumes only a finite number of observations on consumer behavior is available. Since all existing data on consumer behavior does consist of finite number of observations, the latter assumption is much more realistic. (Varian 1983, p. 99)

We can explain the non-parametric method by means of the concept of data structure (or, as Suppes prefers to call it, model of data). Observing the behavior of the agent we get the structure of data:

$$\hat{\mathfrak{D}} = \langle X, F, \hat{\eta} \rangle,$$

where $\hat{\eta}$ is a función defined over a finite subset F of $P \times W$, precisely that of pairs of systems of prices and income levels under which the behavior of the consumer has been observed. The empirical (actually observed) demand function $\hat{\eta}$ is obviously finite and discrete, and that is why structure $\mathfrak{D}$ can be depicted by means of the table.

The non-parametric method does not try to build directly the complete demand function η, but rather uses an algorithm to build directly a utility function that "rationalizes" the data, provided that structure $\mathfrak{D}$ satisfies the conditions of the following theorem.

THEOREM. (AFRIAT) *The following conditions are equivalent:*

(1) There exists a non-satiated utility function that rationalizes the data.

(2) The data satisfy the general axiom of revealed preference.

(3) There are numbers U^i, $\lambda^i > 0$ $(i = 1, \ldots, N)$ that satisfy Afriat's inequalities:

$$U^i \leq U^j + \lambda^j \mathbf{p}^j(\mathbf{x}^i - \mathbf{x}^j)$$

for $i, j = 1, \ldots, N$.

(4) There is a concave, monotonic, continuous and non-satiated utility function that rationalizes the data.[4]

Afriat (1967) and Varian (1982) provided algorithms by means of which it is possible to build a utility function u that rationalizes the data of structure $\mathfrak{D}$. This is "jumping", so to say, from a model of data to the theoretic function u. Then it is possible to imbed $\mathfrak{D}$ in the partial structure consisting of the Walrasian demand function μ induced by u. That is to say, μ restricted to F coincides with $\hat{\eta}$. The parametric method, on the other hand, tries first to determine the non-theoretic demand function η without involving u and afterward tries to recover u out of η.

The upshot of this example is that there are some applications of economic theories that require methods other than those of RTM in order to flesh out the terms of the theory, to actually measure the magnitudes referred to by the terms of the theory. Suppes himself used methods more similar to this (fleshing out the magnitudes out of models of data) in his classical experiments on learning theory.[5] Hence, more than a method to

[4] For a proof see Varian (1983).

[5] For a description of these see García de la Sienra 2011.

flesh out the terms of a theory in empirical application or testing, in economics RTM seems to be rather a methodology to establish the existence of metrizations, sometimes even for idealized magnitudes, and especially for these.

References

Balzer, W., C.U. Moulines, and J.D. Sneed, 1987, *An Architectonic for Science. The Structuralist Program*, D. Reidel Publishing Company, Dordrecht.

Balzer, W., J.D. Sneed, and C.U. Moulines (eds.), 2000, *Structuralist Knowledge Representation. Paradigmatic Examples* (*Poznań Studies in the Methodology of the Sciences and the Humanities*, vol. 75), Rodopi, Amsterdam.

Beaver, W.H., and J.S. Demski, 1979, "The Nature of Income Measurement", *The Accounting Review*, vol. 54, no. 1, pp. 38–46.

Boumans, M., 2007a, *Measurement in Economics. A Handbook*, Elsevier, Amsterdam.

—, 2007b, "Introduction", in Boumans 2007a, pp. 3–18.

—, 2008, "Measurement", in Durlauf and Blume 2008.

Brown, S., 1984, *Leibniz*, University of Minnesota Press, Minneapolis.

Clarke, B.L, 1985, "Individuals and Points", *Notre Dame Journal of Formal Logic*, vol. 26, no. 1, pp. 61–75.

Díez, J.A., 2000, "Structuralist Analysis of Theories of Fundamental Measurement", in Balzer, Sneed and Moulines 2000, pp. 19–49.

Durlauf, S.N. and L.E. Blume (eds.), 2008, *The New Palgrave Dictionary of Economics*, second edition, Palgrave Macmillan, London.

García de la Sienra, A., 2011, "Suppes' Methodology of Economics", *Theoria*, vol. 26, no. 3, pp. 347–366.

Krantz, D.H., R.D. Luce, P. Suppes and A. Tversky, 2007, *Foundations of Measurement*, 3 vols., Dover, Mineola, NY.

Luce, R.D. and P. Suppes, 2002, "Representational Measurement Theory", in Wixted and Pashler 2001, pp. 1–41.

Michell, J., 2007, "Representational Theory of Measurement", in Boumans 2007, pp. 19–39.

Morgan, Mary S., 2007, "An Analytical History of Measuring Practices: The Case of Velocities of Money", in Boumans 2007, pp. 105–132.

Parkinson, G.H.R. (ed.), 1981, *Leibniz: Philosophical Writings*, J.M. Dent and Sons, London.

Suppes, P., 2002, *Representation and Invariance of Scientific Structures*, CSLI Publications, Stanford.

Walras, L. (1954). *Elements of Pure Economics*. London: Allen and Unwin.

Wixted, J., and H. Pashler (eds.), 2002, *Steven's Handbook of Experimental Psychology*, vol. 4, Wiley, New York.