Approximations for the period of the simple pendulum based on the arithmetic-geometric mean

Claudio G. Carvalhaes[a)]
Center for the Study of Language and Information, Ventura Hall, 210 Panama Street, Stanford University, Stanford, California, 94305-4101 and Instituto de Matemática e Estatística, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, RJ, 20550-900, Brazil

Patrick Suppes[b)]
Center for the Study of Language and Information, Ventura Hall, 210 Panama Street, Stanford University, Stanford, California, 94305-4101

We use the arithmetic-geometric mean to derive approximate solutions for the period of the simple pendulum. The fast convergence of the arithmetic-geometric mean yields accurate solutions. We also discuss the invention of the pendulum clock by Christiaan Huygens in 1656–1657. © *2008 American Association of Physics Teachers.*
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I. INTRODUCTION

The period of the simple pendulum can be expressed as a nonelementary function of its amplitude and cannot be evaluated analytically without approximation. The linearization of the equation of motion gives an approximate solution for small oscillations.[1] In this regime the pendulum behaves like a harmonic oscillator and is a classic example of isochronism. For larger oscillations the pendulum's motion is anisochronous, and the linear equation does not describe the oscillations. The lack of an elementary closed-form restricts the study of the simple pendulum at the undergraduate level to small oscillations. Approaches based on the expansion of the period in terms of a power series of the amplitude are sometimes presented in advanced courses,[2] but converge slowly even for intermediate amplitudes. Several alternatives have been proposed[3–13] that differ in complexity and domains of validity; some are intuitive, and others are based on relatively sophisticated procedures.

In this paper we study a sequence of approximations for the pendulum period based on the use of the arithmetic-geometric mean. This quantity has been used to approximate the complete elliptic integrals of the first and the second kind with high accuracy.[14–18] The idea is to compute a sequence of arithmetic and geometric means of two positive numbers which converge toward each other as more iterations are performed. The convergence is proportional to the reciprocal of the complete elliptic integral of the first kind, and so the sequence can be used to accurately determine the pendulum period for arbitrary amplitudes.[19,20] For instance, convergence with eight digits of accuracy is obtained for an amplitude of 179° after only four iterations.

If the arithmetic-geometric mean's iterations are calculated analytically, the successive means provide a sequence of analytical approximations for the period. The relation between two successive iterations of this sequence has a clear physics interpretation in terms of a renormalization process in which a given pendulum is replaced by another of the same period but of smaller amplitude, until the iteration reaches a sufficiently small amplitude that the calculation of the period is accurately handled by a simple approximation.

Section II reviews the formulation of the period of the nonlinear pendulum in terms of the complete elliptic integral of the first kind. In Sec. III we define the arithmetic-geometric mean, illustrate its calculation, and show how it is related to the pendulum period. We also highlight the numerical constraints on the computation of the exact period. In Sec. IV we present approximate solutions for the period based on the arithmetic-geometric mean and compare them to other published solutions. After our conclusions in Sec. V, we include in Sec. VI a historical note on the invention of the pendulum clock by Christiaan Huygens.

II. THE PENDULUM PERIOD

A simple pendulum consists of a particle of mass m hanging from an unstretchable, rigid massless rod of length L fixed at a pivot point. The system freely oscillates in a vertical plane under the action of gravity. It is assumed that the motion is not affected by damping or external forcing. For the initial conditions $\theta(0) = \theta_0$ and $d\theta/dt(0) = 0$, with the zero of the potential energy at the bottom of the pendulum's trajectory as shown in Fig. 1, the equation of energy conservation is

$$mgL(1 - \cos \theta_0) = \frac{1}{2}mgL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1 - \cos \theta), \quad (1)$$

where g is the local acceleration of gravity. We solve Eq. (1) for the angular velocity $d\theta/dt$ in the first quarter of its period and obtain[13]

$$T = T_0 \frac{\sqrt{2}}{\pi} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}}, \quad (2)$$

where $T_0 = 2\pi\sqrt{L/g}$ is the period in the small-angle approximation. The improper integral in Eq. (2) is cast into a more convenient form by using the identity $\cos \theta = 1 - 2\sin^2(\theta/2)$ and the change of variable $\sin \phi = \sin(\theta/2)/\sin(\theta_0/2)$, which leads to

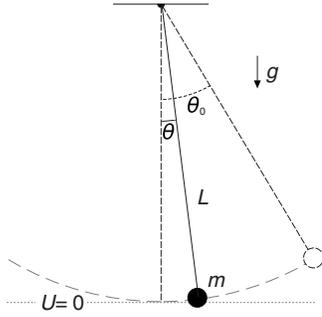$$T = \frac{2}{\pi}K(k)T_0, \quad (3)$$

where $k = \sin(\theta_0/2)$, and

Fig. 1. A simple pendulum of mass m and length L oscillating in a vertical plane. The particle is displaced to the angle θ_0 and then released at rest. The system oscillates under the action of gravity. The angular displacement θ is measured counter-clockwise relative to the equilibrium position ($\theta=0$).

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad (|k| < 1 \text{ and } |\theta_0| < 180^\circ). \tag{4}$$

The integral in Eq. (4) is called a complete elliptic integral of the first kind. For small amplitudes ($k \approx 0$), K converges to $\pi/2$ so that the amplitude-independent approximation T_0 readily emerges from Eq. (3). As shown in Fig. 2, K is a monotonically increasing function, so that T_0 underestimates T with increasing inaccuracy. The range in which T_0 is a valid approximation for T is somewhat arbitrary, but some authors[21] consider it to correspond to the largest interval for which T_0 differs from the exact value of T by less than 1%, which is roughly $|\theta_0| < 23^\circ$.

III. THE ARITHMETIC-GEOMETRIC MEAN

The function $K(k)$ cannot be expressed in terms of elementary functions, but can be numerically evaluated with high precision by a simple procedure. To do so we first introduce the two-term recursion relation

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}), \tag{5a}$$

$$b_n = \sqrt{a_{n-1} b_{n-1}}, \tag{5b}$$

where a and b are real numbers such that $0 \le b < a$ with $a_0 = a$ and $b_0 = b$. It can be shown that $\sqrt{a_n} \ge \sqrt{b_n}$ for all $n \ge 1$, and hence,[14,17,18]
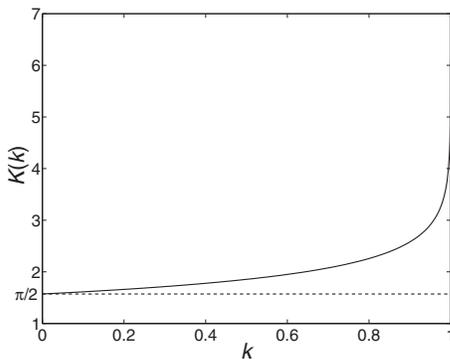

Fig. 2. The complete elliptic integral of the first kind. The function has a global minimum of $\pi/2$ at $k=0$ ($\theta_0=0^\circ$) and monotonically increases to infinity at $k=1$ ($\theta_0=180^\circ$). The dashed line corresponds to the small-angle approximation $K(k)=\pi/2$.

Table I. The first four iterations of the arithmetic-geometric mean for $a=1$ and $b=0.8$. The last column shows the number of digits to the right of the decimal point for which the means match each other.

Iteration	Mean		
n	Arithmetic (a_n)	Geometric (b_n)	Digits of accuracy
1	0.9000000000000000	0.8944271909999159	0
2	0.8972135954999579	0.8972092687327323	4
3	0.8972114321163451	0.8972114321137369	11
4	0.8972114321150411	0.8972114321150410	15

$$a_n - b_n \le \frac{1}{2}(a_{n-1} - b_{n-1}) \le \cdots \le \frac{1}{2^n}(a - b). \tag{6}$$

Because the right-hand side of this inequality goes to zero as $n \to \infty$, the sequences a_n and b_n converge to a common limit uniquely determined by a_0 and b_0. This limit is denoted $M(a,b)$ and is called the arithmetic-geometric mean of a and b,

$$M(a,b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n. \tag{7}$$

Table I illustrates the solution of the recursion relations (5) for $(a,b)=(1,0.8)$. Agreement of 15 decimal places between the means is obtained after only four iterations, suggesting that the arithmetic-geometric mean rapidly converges. Because the convergence of the arithmetic-geometric means is quadratic, an agreement of about 2^n digits between the means is expected after n iterations. To see so we introduce a measure of the error at the nth iteration

$$c_n = \sqrt{a_n^2 - b_n^2}. \tag{8}$$

We observe that

$$c_{n+1} = \frac{1}{2}(a_n - b_n) = \frac{c_n^2}{4a_{n+1}}. \tag{9}$$

We see that the sequence c_n goes to 0 quadratically, so that the convergence of the arithmetic-geometric means is second order (quadratic).

We introduce the family of integrals

$$I(a,b) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}, \quad (a,b > 0), \tag{10}$$

which contains K as the special case

$$K(k) = I(1, \sqrt{1 - k^2}). \tag{11}$$

The change of variable $t = y \tan \phi$ followed by $u = (t - ab/t)/2$ and some rearrangements in Eq. (11) leads to the key relation

$$I(a,b) = I(a_1, b_1), \tag{12}$$

where a_1 and b_1 are the arithmetic and geometric means of a and b, respectively. Equation (12) shows that I is invariant under the mapping $(a,b) \to (a_1, b_1)$. Successive applications of this property yield

$$I(a,b) = I(a_n, b_n), \tag{13}$$

and I is invariant under the arithmetic-geometric mean recursion relation in Eq. (5). If we take the limit $n \to \infty$ in Eq. (13)

and use the continuity of $I(a,b)$ and the definition Eq. (7), we obtain

$$I(a,b) = \lim_{n \to \infty} I(a_n, b_n)$$

$$= I(M(a,b), M(a,b)) = \int_0^{\pi/2} \frac{d\phi}{M(a,b)} = \frac{\pi}{2M(a,b)}. \tag{14}$$

It follows from Eq. (11) that

$$K(k) = \frac{\pi}{2M(1, \sqrt{1-k^2})} = \frac{\pi}{2a_\infty}, \tag{15}$$

where $a_\infty = \lim_{n \to \infty} a_n$. We use this result in Eq. (3) to infer that

$$T = \frac{T_0}{a_\infty}. \tag{16}$$

Equation (16) yields the exact solution of the pendulum period in terms of the arithmetic-geometric mean. It also represents an iterative algorithm for determining T numerically for arbitrary amplitudes. The only constraint on this computation is the finite representation of real numbers in a computer. Our computations used the IEEE floating-point standard,[22] which defines the distance from 1.0 to the next largest 64-bit double format number as $\epsilon = 2^{-52}$. This number is called the machine epsilon or the unit round-off. For this criterion the iterations cease at the first step for which the error c_n given by Eq. (8) satisfies $c_n < \epsilon$. The number of iterations need to achieve this precision depends on the amplitude. The quadratic convergence of the arithmetic-geometric means ensures that only a few iterations are necessary, even for very large amplitudes. This statement is confirmed by Table II, which shows that only seven itera-

Table II. Maximum range of the amplitude for which the period obtained with n iterations of the arithmetic-geometric mean differs from the exact value by less than $\epsilon = 2^{-52}$ (A_1) or by less than 1% (A_2). Units were chosen such that $T_0 = 1$.

n	A_1	A_2
1	0.04°	96.11°
2	4.57°	163.10°
3	45.39°	179.37°
4	126.17°	179.99°
5	173.21°	179.99°
6	179.88°	179.99°
7	179.99°	179.99°

tions are sufficient to obtain the exact solution to machine precision for a wide range of amplitudes. For the interval $|\theta_0| < 179°$ the number of required iterations reduces to 3 to achieve agreement within 1% with the exact value. The computation of T in this way is easily performed by using a spreadsheet or a pocket calculator.

IV. ARITHMETIC-GEOMETRIC-MEAN BASED APPROXIMATIONS FOR THE PENDULUM PERIOD

If the calculations of the recursion relations of Eq. (16) are performed analytically, a sequence $\{T_n\}$ of algebraic expressions is obtained which converges to the exact solution in the ranges shown in Table II. The elements of $\{T_n\}$ are given by

$$T_n = \frac{T_0}{a_n}, \tag{17}$$

where a_n is the arithmetic mean at the nth iteration in the calculation of $M(1, \sqrt{1-k^2})$. Let $q = \sqrt{1-k^2} = \cos(\theta_0/2)$. Then the first four iterations give

$$T_1 = \frac{2T_0}{1+q}, \tag{18a}$$

$$T_2 = \frac{4T_0}{1 + q + 2q^{1/2}}, \tag{18b}$$

$$T_3 = \frac{8T_0}{1 + q + 2q^{1/2} + 2^{3/2}q^{1/4}(1+q)^{1/2}}, \tag{18c}$$

$$T_4 = \frac{16T_0}{1 + q + 2q^{1/2} + 2^{3/2}q^{1/4}(1+q)^{1/2} + 2^{7/4}q^{1/8}(1+q)^{1/4}(1+q+2q^{1/2})^{1/2}}. \tag{18d}$$

The approximation $T_n \approx T_0$ for small amplitudes is recovered from Eq. (18) in the limit $q = 1$ ($\theta_0 = 0$). The approximation $T_0(1 + \theta_0^2/16)$, often cited as the first correction to T_0, emerges from the expansion of T_1 in θ_0, truncated at first order. For the upper limit $\theta_0 = 180°$ ($q = 0$) we obtain the sequence $\{2T_0, 4T_0, 8T_0, 16T_0, \ldots\} = \{2^n T_0\}$, which shows the exponential divergence of the exact solution T_∞.

Equation (18) represent a good approximation for the pe-

riod. From Table II we see that seven iterations were needed to achieve a result for the period for $|\theta_0| < 179.99°$ with machine epsilon precision. Further computations were performed with the aid of a symbolic manipulation package. Because the analytical expressions for T_5, T_6, and T_7 are too long, they are have not given here. The expression for T_{n+1} can be obtained from T_n by replacing q by $2\sqrt{q}/(1+q)$ (the harmonic mean of 1 and q) and replacing T_0 by $2T_0/(1+q)$,

Table III. Approximate expressions for the period of the simple pendulum given by various authors. T_0 was taken equal to one and A_1 and A_2 are as in Table II.

Reference	Approximation	A_1	A_2
3	$1 + \theta_0/16$	0.03°	73.91°
5	$[(\sqrt{3}\theta_0/2)/\sin(\sqrt{3}\theta_0/2)]^{1/2}$	0.08°	135.07°
6	$1 - \theta_0^2/8$	0.03°	78.50°
7	$(\theta_0/\sin\theta_0)^{3/8}$	0.05°	111.87°
8	$1/\sqrt{q}$	0.04°	95.79°
11	$(\sqrt{q})^{-\sqrt[8]{q}}$	1.11°	177.15°
12	$[\theta_0/2J_1(\theta_0)]^{1/2}$	0.05°	116.56°
13	$\log q/(q-1)$	0.05°	120.54°

$$T_{n+1}(q, T_0) = T_n\left(\frac{2\sqrt{q}}{1+q}, \frac{2T_0}{1+q}\right). \tag{19}$$

Equation (19) provides an alternative procedure for computing the arithmetic-geometric mean approximations recursively and is suitable for computer algebra implementations. It also allows us to establish the arithmetic-geometric mean as a renormalization process[19] in which the pendulum is recursively replaced by another of smaller amplitude but of the same period. Let $q_{n+1} = 2\sqrt{q_n}/(1+q_n)$ and $\tau_{n+1} = 2\tau_n/(1+q_n)$, with $(q_1, \tau_1) = (q, T_0)$. We infer from Eq. (19) that

$$T_n = T_n(q, T_0) = T_n(q_1, \tau_1) = T_{n-1}(q_2, \tau_2) = \cdots = T_1(q_n, \tau_n). \tag{20}$$

For convenience, we associate each pair (q_n, τ_n) with an effective amplitude θ_n and an effective length L_n to obtain

$$T_n = T_1(\theta_n, L_n), \tag{21}$$

where $q_n = \cos\theta_n/2$, $\tau_n = 2\pi\sqrt{L_n/g}$, and $(\theta_1, L_1) = (\theta_0, L)$. It follows from the inequalities $q_n > q_{n-1} > \cdots > q$ and $\tau_n > \tau_{n-1} > \cdots > T_0$ that $\theta_n < \theta_{n-1} < \cdots < \theta_0$ and $L_n > L_{n-1} > \cdots > L$. From these inequalities we see from Eq. (21) that the arithmetic-geometric mean algorithm preserves the period and recursively decreases the pendulum's amplitude until T_1 becomes sufficiently accurate. According to Table II the recursions on the argument of T_1 will proceed until $\theta_n \leq 0.04°$ for ϵ precision. The number of iterations required for this computation depends only on the amplitude.

Table III shows the expressions of other approximations achieved by a variety of other methods and gives their agreement with the exact result. Comparing the ranges of Tables II and III, we observe that the arithmetic-geometric mean solutions are much more accurate than that given by the other methods. Among these solutions the solution proposed by Hite[11] is the most satisfactory. It outperforms T_2 for amplitudes greater than 163°. This approximation suddenly fails to fulfill the requirements of being increasing and concave up for greater amplitudes, for which it is outperformed by T_3 whose relative error is less than 1% for $|\theta_0| < 179°$ and agrees with the exact solution in the interval $|\theta_0| < 45°$ with machine-epsilon precision.

V. CONCLUSION

The arithmetic-geometric mean algorithm was used to derive a sequence of approximate solutions for the period of the simple pendulum. This sequence quickly converges to the exact solution and provides expressions which are much more accurate than those in the literature. Although power-series expansions can be used to achieve similar results, many more than a hundred terms must be included, depending on the amplitude. For instance, for $|\theta_0| = 179°$ truncation at order 500 leads to an approximation which differs from the exact solution by more than 20%.[20]

The approach discussed here is not new, but is not widely known. Due to the relevance of elliptic integrals in physics, the approach is useful in many other applications.[23,24] Because the arithmetic-geometric mean operates by renormalizing the pendulum's amplitude, it offers a good physics framework for introducing the concept of renormalization.

The arithmetic-geometric mean is accessible to intermediate level students and is easy to implement in a spreadsheet. For this reason this approach can be of value for instructors working with spreadsheet-based activities.[25–27]

VI. HISTORICAL NOTE

The pendulum clock was invented by Christiaan Huygens in the winter of 1656–1657. In 1673 Huygens published his magnum opus.[28] It is now generally recognized that, after Newton's 1687 *Principia*, Huygens' treatise was the most original work in mathematical physics in the 17th century. An early anticipation of the principle of conservation of energy is stated as the single new Hypothesis or Axiom required for determining the center of oscillation.

> *If any number of weights begin to move by the force of their own gravity, their center of gravity cannot rise higher than the place at which it was located at the beginning of the quote* (Ref. 28, p. 108).

At the end of the treatise, Huygens reaches a result equivalent to the result for small angles $T = 2\pi\sqrt{L/g}$.

Huygens had no knowledge of the arithmetic-geometric mean, which was first analyzed in a paper by Lagrange (1784–85),[29] and much studied by Gauss as a young man in unpublished papers, the one on arithmetic-geometric mean having been written in 1799 or 1800.[30]

a)Electronic mail: claudioc@stanford.edu

b)Electronic mail: psuppes@stanford.edu

[1] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 7th ed. (Wiley, Hoboken, NJ, 2004).

[2] S. Thornton and J. Marion, *Classical Dynamics of Particles and Systems*, 4th ed. (Saunders College, Fort Worth, TX, 1995).

[3] D. S. Mathur, *Elements of Properties of Matter* (S. Chand & Company, New Delhi, 1962).

[4] L. P. Fulcher and B. F. Davis, "Theoretical and experimental study of the motion of the simple pendulum," Am. J. Phys. **44**, 51–55 (1976).

[5] W. P. Ganley, "Simple pendulum approximation," Am. J. Phys. **53**, 73–76 (1985).

[6] L. H. Cadwell and E. R. Boyko, "Linearization of the simple pendulum," Am. J. Phys. **59**, 979–981 (1991).

[7] M. I. Molina, "Simple linearization of the simple pendulum for any amplitude," Phys. Teach. **35**, 489–490 (1997).

[8] R. B. Kidd and S. L. Fogg, "A simple formula for the large-angle pendulum period," Phys. Teach. **40**, 81–83 (2002).

[9] L. E. Millet, "The large-angle pendulum period," Phys. Teach. **41**, 162–163 (2003).

[10] R. R. Parwani, "An approximate expression for the large angle period of a simple pendulum," Eur. J. Phys. **25**, 37–39 (2004).

[11] G. E. Hite, "Approximations for the period of a simple pendulum," Phys. Teach. **43**, 290–292 (2005).

[12] A. Beléndez, T. Hernández, A. Beléndez, and C. Neipp, "Analytical approximations for the period of a nonlinear pendulum," Eur. J. Phys. **27**, 539–551 (2006).

[13] F. M. S. Lima and P. Arun, "An accurate formula for the period of a simple pendulum oscillating beyond the small angle regime," Am. J. Phys. **74**, 892–895 (2006).

[14] M. Abramowitz, *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1974).

[15] E. Salamin, "Computation of π using arithmetic-geometric mean," Math. Comput. **30**, 565–570 (1976).

[16] J. M. Borwein and P. B. Borwein, "The arithmetic-geometric mean and fast computation of elementary functions," SIAM Rev. **26**, 351–366, 1984.

[17] G. Almkvist and B. Berndt, "Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses π, and the ladies diary," Am. Math. Monthly **95**, 585–608 (1988).

[18] G. E. Andrews, R. Askey, and R. Roy, *Special Functions* (Cambridge University Press, New York, 1999).

[19] S. Bullett and J. Stark, "Renormalizing the simple pendulum," SIAM Rev. **35**, 631–640 (1993).

[20] S. Siboni, "Superlinearly convergent homogeneous maps and period of the pendulum," Am. J. Phys. **75**, 368–373 (2007).

[21] N. Aggarwal, N. Verma, and P. Arun, "Simple pendulum revisited," Eur. J. Phys. **26**, 517–523 (2005).

[22] D. Stevenson and others, "An American national standard: IEEE standard for binary float point arithmetic," ACM SIGPLAN Notices **22**, 9–25 (1987).

[23] R. H. Good, "Elliptic integrals, the forgotten functions," Eur. J. Phys. **22**, 119–126 (2001).

[24] R. G. Forbes, "Simple good approximations for the special elliptic functions in standard Fowler–Nordheim tunneling theory for a Schottky–Nordheim barrier," Appl. Phys. Lett. **89**, 113122-1–3 (2006).

[25] B. A. Cooke, "Some ideas for using spreadsheets in physics," Phys. Educ. **32**, 80–87 (1997).

[26] J. E. Baker and S. J. Sugden, "Spreadsheets in education: The first 25 years," Spread. Educ. **1**, 18–43 (2003).

[27] M. Fowler, "Using Excel to simulate pendulum motion and maybe understand calculus a little better," Sci. Educ. **13**, 791–796 (2004).

[28] C. Huygens, *Horologium Oscillatorium sive De Motu Pendulorum ad Horologia Aptato Demonstrationes Geometric*, Paris, 1673; translated into English by R. J. Blackwell as *The Pendulum Clock, or Geometrical Demonstrations Concerning the Motion of Pendula as Applied to Clocks* (Iowa State University Press, Ames, 1986).

[29] J.-L. Lagrange, "Sur une nouvelle méthode de calcul intégral pour les différentielles affectées d'un radical carré sous lequel la variable ne passe pas le quatrième degré," Mem. l'Acad. Roy. Sci. Turin, 2 (1784–85); Reprinted in *Oeuvres de Lagrange, tome 2* (Gauthier-Villars, Paris, 1868), pp. 251–312.

[30] K. F. Gauss, Arithmetisch Geometrisches Mittel. *Werke* 3, 361–432 (1799) (K. Gesellschaft der Wissenschaften zu Göttingen, Leipzig-Berlin, 1866).