We want to begin this paper with recording our joint indebtedness to de Finetti for so enriching the theory of probability, especially its foundations.

Suppes:
I met de Finetti in May of 1960 at a symposium in Paris on decision theory. We had several lively informal conversations about the role of the axiom of choice in the foundations of probability, especially in relation to the existence of countably additive measures. Over the years I had the opportunity to meet de Finetti on various occasions. I recall a memorable walk with de Finetti and Jimmy Savage in the gardens of the Villa Frascati near Rome later in the 1960s. On this occasion we had a long discussion of determinism and quantum mechanics, and what the existence of indeterminism and quantum mechanics implied for subjective theories of probability. The last time I saw de Finetti was in Rome in March 1979. My wife and I invited de Finetti to dinner, and after dinner he and I had a wide-ranging philosophical discussion, which he conducted with his usual vigor. In fact, even though he was then in his late seventies, I finally gave up at about 12:30 a.m., and said good night, outdone by his continuing vitality and energy in philosophical conversation.

Zanotti:
I met de Finetti in 1971 at de Finetti’s summer residence in the vicinity of Rome, after correspondence beginning in 1969. He stated in one of the letters and in discussion his sympathy for the main aspects of Quine’s philosophy. I especially appreciate his subjective but pluralistic view towards the foundations and application of probability, as opposed to the rigidity and limitations of many objective theories of probability. All of my thinking about probability has been strongly influenced by de Finetti.

In this paper we prove four theorems about the existence of a probability measure when a pair of upper and lower probabilities satisfy certain conditions. De Finetti did not raise this problem in precisely the form we are stating it here, but in the final appendix to his two-volume treatise, *Theory of Probability* (de Finetti 1975, Volume 2, Appendix, Section 19.3), he discusses the question of whether imprecise probabilities exist, and he gives several different examples in which one may want to express oneself in terms of an imprecise
evaluation, as expressed by an upper and lower probability rather than a probability itself. In this section and as far as we have been able to determine elsewhere in the treatise as well, he does not raise, however, the question of whether, given a pair of upper and lower probabilities, does a probability exist that is bounded by the upper and lower probabilities. His discussion in Section 19.3 of the Appendix implicitly assumes that such a probability always does exist. It is the purpose of our paper to show some conditions under which a probability does exist. In the final theorem of the four we are also concerned with under what conditions a unique probability measure can be said to generate the upper and lower probability measures. This last theorem is related to the earlier work of Dempster (1967).

The results in this paper extend and go beyond those given in our earlier paper on using random relations to generate upper and lower probabilities (Suppes and Zanotti 1977). On the other hand, an essential tool of the mathematical constructions used in the present paper we used in two earlier papers on proving under what conditions a qualitative probability relation or a conditional probability qualitative relation has a unique numerical representation by a probability measure (Suppes and Zanotti 1976, 1982). The device was to move from the Boolean algebra of events to the semigroup of extended indicator functions generated by these events. The application is different here but it is apparent, from the rather different kinds of results obtained earlier, that the semigroup of extended indicator functions is a useful structure for studying various foundational questions in probability (the formal definitions of these concepts are given below).

1. THE FIRST THEOREM

The first theorem we prove is related to a theorem proved by Dana Scott (1964), where he gives what is now his well-known result on necessary and sufficient conditions for a qualitative ordering on the Boolean algebra of a finite set to have a representation by a probability measure. To accentuate the four main theorems we label all other statements that are proved as lemmas, propositions, or corollaries. We use repeatedly the abbreviation iff for if and only if.

DEFINITION 1. Let \( \Omega \) be a set and \( \mathcal{B} \) a Boolean algebra of subsets
of $\Omega$. A pair of functions $P_*: \mathcal{B} \to [0, 1]$ and $P^*: \mathcal{B} \to [0, 1]$ is an upper-lower functional on $(\Omega, \mathcal{B})$ iff the pair satisfies the following properties for all $A$ in $\mathcal{B}$:

(i) $P_*(\emptyset) = P^*(\emptyset) = 0$,

where $\emptyset$ is the empty set,

(ii) $P_*(\Omega) = P^*(\Omega) = 1$,

(iii) $P_*(A) \leq P^*(A)$.

We write $G(\mathcal{B})$ for the additive semigroup generated by the indicator functions of the subsets of $\Omega$ in $\mathcal{B}$. The elements of the additive semigroup are then just what we have called earlier extended indicator functions, that is, functions that are finite sums of indicator functions. We shall also follow the notation used in our earlier papers referred to above and denote the indicator function of a set $A$ by $A^c$. In what follows we shall be almost entirely concerned with finite families of sets and finite partitions. This finite restriction is in the spirit of de Finetti's concentration on finite additivity. It is apparent, but we want to emphasize that the notation $\sum A_i^c$ is a notation for a (finite) sum of indicator functions, and such a sum is of course an extended indicator function.

In the first lemma we use the following compact notation:

$$
(\bigcup A_i)^c = 1 \land (\sum A_i^c) = \{\omega \in \Omega: \sum A_i^c(\omega) \geq 1\}^c \\
(\bigcup (A_i \cap A_j))^c = 2 \land (\sum A_i^c) = \{\omega \in \Omega: \sum A_i^c(\omega) \geq 2\}^c \\
\ldots \\
(\bigcup A_i)^c = n \land (\sum A_i^c) = \{\omega \in \Omega: \sum A_i^c(\omega) \geq n\}^c,
$$

where the $\sum, \bigcup, \land \cap$ are for $i = 1, \ldots, n, i, j = 1, \ldots, n, \text{etc.}$

**LEMMA 1.** For any two finite families of sets $\{A_i\}$ and $\{B_j\}$,

$$
\sum A_i^c = \sum B_i^c \text{ iff for } j \geq 1, j \land \sum A_i^c = j \land \sum B_i^c.
$$

Hereafter we drop the summation indices $m$ and $n$, which will be understood. Note that it is convenient to talk about representations of an extended indicator function in terms of the summation of what can be different indicator functions. It is obvious that in general an extended indicator function can be represented in many different ways
as a sum of indicator functions. We call a particular family a representation.

COROLLARY. \( A^c = \sum B_i^c \) iff \( \{B_i\} \) is a finite partition of \( A \).

Given an upper-lower functional \( (P_*, P^*) \) on \( (\Omega, \mathcal{B}) \) we may define the following pair of functionals \( (F_*, F^*) \) on the additive semigroup \( G(\mathcal{B}) \):

\[
F_*(f) = \text{SUP}\left\{ \sum P_*(E_i) : \sum E_i = f \right\} \\
F^*(f) = \text{INF}\left\{ \sum P^*(E_i) : \sum E_i = f \right\}
\]

for \( f \) in \( G(\mathcal{B}) \), where SUP and INF are taken over all possible representations of \( f \). Elementary properties of the pair \( (F_*, F^*) \) are stated in the next lemma.

LEMMA 2. The pair \((F_*, F^*)\) satisfies the following properties for all \(f, g\) in \( G(\mathcal{B}) \) and \( E \) in \( \mathcal{B} \):

1. \( 0 \leq F_*(f) < \infty \),
2. \( F_*(f + g) \geq F_*(f) + F_*(g) \),
3. \( P_*(E) \leq F_*(E) \),
4. \( 0 \leq F^*(f) < \infty \),
5. \( F^*(f + g) \leq F^*(f) + F^*(g) \),
6. \( P^*(E) \geq F^*(E) \).

**Proof.** Straightforward.

LEMMA 3. The restriction of \((F_*, F^*)\) to \( \mathcal{B} \) will be \((P_*, P^*)\) iff for all \(A, B\) in \( \mathcal{B} \) with \( A \cap B = \phi \) we have:

(i) \( P_*(A \cup B) \geq P_*(A) + P_*(B) \),
(ii) \( P^*(A \cup B) \leq P^*(A) + P^*(B) \).

**Proof.** Clearly, \( P_*(E) \leq F_*(E) \). If \( \sum B_i^c = E^c \), then from the corollary \( \{B_i\} \) is a partition of \( E \). By (i) we have \( P_*(E) \geq \sum P_*(B_i) \), hence

\[
P_*(E) \geq \text{SUP}\{\sum P_*(B_i) : \sum B_i^c = E^c\} = F_*(E^c).
\]

Conversely, from \( P_*(E) = F_*(E^c) = \text{SUP}\{\sum P_*(B_i) : \{B_i\} \text{ a partition of} \}

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$E \{ we have $ P_*(E) \geq \sum P_*(B_i) $ for each partition $ \{B_i\} $ of $ E $. Similarly for $ P^* $.

**LEMMA 4.** Given an upper-lower functional $(P_*, P^*)$, $ F_*(f) \leq F^*(f) $ iff for any two representations of extended indicator function $ f = \sum A_i^c = \sum B_i^c $, we have

$$ \sum P_*(A_i) \leq \sum P^*(B_i). $$

**Proof.** Suppose $ F_*(f) \leq F^*(f) $. Let $ f = \sum A_i^c = \sum B_i^c $. Then we have

$$ \sum P_*(A_i) \leq F_*(f) \leq F^*(f) \leq \sum P^*(B_i). $$

Conversely, suppose that for any two representations $ \sum A_i^c = \sum B_i^c = f, \sum P_*(A_i) \leq \sum P^*(B_i) $. Then by letting SUP and INF range over all representations we have:

$$ F_*(f) = \text{SUP} \left\{ \sum P_*(A_i) : \sum A_i^c = f \right\} $$

$$ \leq \text{INF} \left\{ \sum P^*(B_i) : \sum B_i^c = f \right\} = F^*(f). $$

We now define the important concept of a separating probability.

**DEFINITION 2.** A probability $ P $ separates an upper-lower functional $(P_*, P^*)$ on $(\Omega, \mathcal{B})$ iff for any $ A $ in $ \mathcal{B} $, $ P_*(A) \leq P(A) \leq P^*(A) $.

The next lemma simply restates a theorem of P. Kranz, which is formulated for arbitrary semigroups. As might be expected, the proof of this lemma depends on the Hahn–Banach theorem.

**LEMMA 5.** (P. Kranz, 1972): Let on a semigroup $ S $ be defined two real functionals $ U $ and $ L : S \to [-\infty, \infty] $ at least one of which is finite, and such that

(i) $ U(s + r) \leq U(s) + U(r) $ for $ s, r $ in $ S $,

(ii) $ L(s + r) \geq L(s) + L(r) $ for $ s, r $ in $ S $,

(iii) $ L(s) \leq U(s) $, $ s $ in $ S $.

Then there exists an additive functional $ \xi $ on $ S $ such that
We are now in a position to state and prove Theorem 1.

**Theorem 1.** There exists a probability $P$ separating an upper-lower functional $(P_*, P^*)$ iff for any two representations $f = \sum A_i^c$ and $f = \sum B_i^c$,

$$\sum P_*(A_i) \leq \sum P^*(B_i).$$

**Proof.** For the necessity, suppose $P$ separates $(P_*, P^*)$. Let $f = \sum A_i^c$ and $f = \sum B_i^c$. Then (by the uniqueness of expectation of indicator functions)

$$\sum P_*(A_i) \leq \sum P(A_i) = \sum P(B_i) \leq \sum P^*(B_i).$$

The sufficiency follows from Lemma 5, a version of the Hahn-Banach theorem for abelian semigroups. From Lemma 4 we have $F_*(f) \leq F^*(f)$ and from Lemma 2, $F_*$ is superadditive and $F^*$ is subadditive. By Lemma 5 there is an additive functional $F$ on $G(\mathcal{B})$ such that

$$F_*(f) \leq F(f) \leq F^*(f)$$

for all $f$ in $G(\mathcal{B})$.

It remains to observe that $F$ restricted to $\mathcal{B}$ is a probability. Since $F_*(\Omega^c) = F^*(\Omega^c) = 1$ and $F_*(\phi^c) = F^*(\phi^c) = 0$, we have $F(\Omega^c) = 1$ and $F(\phi^c) = 0$. If $A$ and $B$ are disjoint sets, then since $(A \cup B)^c = A^c + B^c$ we have $F((A \cup B)^c) = F(A^c) + F(B^c)$.

The form of Kranz's theorem (Lemma 5) suggests that a sufficient condition for a separating probability would be the condition: if $A \cap B = \phi$, then

$$P_*(A) + P_*(B) \leq P_*(A \cup B) \leq P^*(A \cup B) \leq P^*(A) + P^*(B),$$

but even with the additional constraint that $P^*(A) = 1 - P_*(\bar{A})$, a counterexample consisting of a Boolean algebra with seven atoms has been given by Walley (1981) and generalized by Papamarcou and Fine (1986).

### 2. The Second Theorem

The second theorem is a characterization in terms of the INF and SUP of a nonempty family of probability measures. Notice that in this case we in general get an entire family of separating probability measures.
DEFINITION 3. \((P_*, P^*)\) is an \((\text{upper-lower})\) envelope of probabilities iff there is a nonempty family \(\mathcal{P}\) of probability measures on \((\Omega, \mathcal{B})\) such that for each \(A\) in \(\mathcal{B}\)

\[
P_*(A) = \inf\{P(A) : P \in \mathcal{P}\}
\]

and

\[
P^*(A) = \sup\{P(A) : P \in \mathcal{P}\}.
\]

THEOREM 2. \((P_*, P^*)\) is an envelope of probabilities iff for any pair of representations \(f = \sum n_i A_i^c\) and \(f = \sum m_i B_i^c\) we have

(i) \[
\sum n_i P_*(A_i) \leq \max_j m_j P^*(B_{\pi(j)}) + \sum_{i \neq j} m_{\pi(i)} P^*(B_{\pi(i)}),
\]

(ii) \[
\sum n_i P^*(A_i) \geq \min_j m_j P_*(B_{\pi(j)}) + \sum_{i \neq j} m_{\pi(i)} P_*(B_{\pi(i)}),
\]

where \(\pi\) is any permutation of the indices.

Proof. Suppose \((P_*, P^*)\) satisfies (i) and (ii). For \(C\) in \(\mathcal{B}\) define \(Q_*(A) = P_*(A)\) if \(A\) is in \(\mathcal{B} - \{C\}\) else \(P^*(C)\). The pair \((Q_*, P^*)\) is an upper-lower functional with \(P_* \preceq Q_* \preceq P^*\). Furthermore, for \(f = \sum A_i^c = nC^c + \sum B_i^c\), with \(B_i \neq C\) we have by (ii):

\[
n P^*(C) + \sum P_*(B_i) \leq \sum P^*(A_i),
\]

that is,

\[
n Q_*(C) + \sum Q_*(B_i) \leq \sum P^*(A_i).
\]

Then by Theorem 1 there exists a probability measure \(P\) on \((\Omega, \mathcal{B})\) with:

\[
P_*(A) \leq Q_*(A) \leq P(A) \leq P^*(A) \quad \text{for all } A \text{ in } \mathcal{B}
\]

and \(P(C) = P^*(C)\). Similarly define \(U^*(A) = P^*(A)\) if \(A\) is in \(\mathcal{B} - \{C\}\) else \(P_*(C)\). The pair \((P_*, U^*)\) is an upper-lower functional with \(P_* \preceq U^* \preceq P^*\). A similar argument using (i) and Theorem 1 produces a probability \(P\) on \((\Omega, \mathcal{B})\) with

\[
P_*(A) \preceq P(A) \preceq P^*(A) \quad \text{for all } A \text{ in } \mathcal{B}
\]

and \(P(C) = P_*(C)\). This process as \(C\) ranges over \(\mathcal{B}\) generates a family \(\mathcal{P}\) of separating probability measures for \((P_*, P^*)\) on \((\Omega, \mathcal{B})\).
The family $\mathcal{P}$ enjoys the following property for all $A$ in $\mathcal{B}$:

$$P_*(A) = \operatorname{INF}\{P(A) : P \in \mathcal{P}\},$$

$$P^*(A) = \operatorname{SUP}\{P(A) : P \in \mathcal{P}\}.$$

This proves sufficiency.

To show condition (i) is necessary, suppose

$$f = \sum n_i A_i^c = \sum m_i B_i^c.$$

For each $P$ in $\mathcal{P}$ and index $j$, we have:

$$\sum n_i P(A_i) \leq m_j P(B_j) + \operatorname{SUP}\left\{ \sum_{i \neq j} m_i P(B_i) : P \in \mathcal{P}\right\}.$$

Taking INF of both sides we obtain:

$$\operatorname{INF}\left\{ \sum n_i P(A_i) : P \in \mathcal{P}\right\} \leq m_j \operatorname{INF}\{P(B_j) : P \in \mathcal{P}\}$$

$$+ \operatorname{SUP}\left\{ \sum_{i \neq j} P(B_i) : P \in \mathcal{P}\right\}.$$

From the subadditivity and superadditivity of SUP and INF, respectively, we have:

$$\sum n_i \operatorname{INF}\{P(A_i) : P \in \mathcal{P}\} \leq m_j \operatorname{INF}\{P(B_j) : P \in \mathcal{P}\}$$

$$+ \sum_{i \neq j} m_i \operatorname{SUP}\{P(B_i) : P \in \mathcal{P}\}$$

or

$$\sum n_i P_*(A_i) \leq m_j P_*(B_j) + \sum_{i \neq j} m_i P^*(B_i),$$

for each index $j$.

The necessity of condition (ii) follows from a similar argument.

It is easy to give an example of an upper-lower functional $(P_*, P^*)$ that is not an envelope of probabilities. Let $A_i, i = 1, \ldots, 4$ be four pairwise disjoint events such that $P_*(A_i) = 0$, and $P_*(A_i \cup A_j) = P^*(A_i \cup A_j) = 1/2$, $i \neq j$. Suppose now that $(P_*, P^*)$ is an envelope. Then there must be a measure $P$ such that $P(A_1) = 0$, $P(A_1 \cup A_2) = 1/2$, so $P(A_2) = 1/2$, and by similar argument $P(A_3) = P(A_4) = 1/2$, so that $P(A_2 \cup A_3 \cup A_4) = 3/2$, which is a contradiction.
3. THE THIRD THEOREM

The third theorem uses the concept of capacity of order two.

DEFINITION 4. Let \( f \in G(\mathcal{B}) \). For each positive integer \( \lambda \), let 
\[
E_\lambda = \{ \omega \in \Omega : f(\omega) \geq \lambda \}. 
\]
(The set \( E_\lambda \) decreases when \( \lambda \) increases.) The representation 
\[
f = \sum E_\lambda 
\]
is the spectral representation of \( f \).

LEMMA 6. Let \( \sum A_i^c \) be any representation of \( f \in G(\mathcal{B}) \). Then 
\[
\left( \bigcup_i A_i \right)^c + \left( \bigcup_{i<j} (A_i \cap A_j) \right)^c + \cdots + \left( \bigcap_i A_i \right)^c
\]
is the spectral representation of \( f \).

Proof. It suffices to observe that 
\[
E_1 = \bigcup_i A_i, E_2 = \bigcup_{i<j} (A_i \cap A_j), \text{ etc.}
\]

DEFINITION 5. The upper-lower functional \( (P_*, P^*) \) being given on 
\( (\Omega, \mathcal{B}) \), the lower integral and upper integral of \( f \in G(\mathcal{B}) \) are the finite 
positive numbers 
\[
\mu_*(f) = \sum_{\lambda > 0} P_*(E_\lambda), \quad \mu^*(f) = \sum_{\lambda > 0} P^*(E_\lambda),
\]
respectively.

LEMMA 7. The pair \( (\mu_*, \mu^*) \) satisfies the following properties for all 
\( f \) in \( G(\mathcal{B}) \) and \( E \) in \( \mathcal{B} \):

1. \( 0 \leq \mu_*(f) < \infty \),
2. \( P_*(E) = \mu_*(E) \),
3. \( 0 \leq \mu^*(f) < \infty \),
4. \( P^*(E) = \mu^*(E) \),
5. \( \mu_*(f) \leq \mu^*(f) \).

DEFINITION 6. An upper-lower functional \( (P_*, P^*) \) on \( (\Omega, \mathcal{B}) \) is a 
capacity of order two iff for all \( A_1 \) and \( A_2 \) in \( \mathcal{B} \)
(i) \[ P_*(A_1 \cup A_2) + P_*(A_1 \cap A_2) \geq P_*(A_1) + P_*(A_2), \]
(ii) \[ P_*(A_1) = 1 - P_*(\overline{A}_1). \]

The concept of capacity has been extensively studied by Choquet (1955).

**Lemma 8.** If the upper-lower functional \((P_*, P^*)\) is a capacity of order two, then for all \(A_1, A_2\) in \(\mathcal{B}\) we have:

(i) \[ P_*(A_1) + P^*(A_2) \geq P_*(A_1 \cup A_2) + P_*(A_1 \cap A_2), \]
(ii) \[ P_*(A_1) + P^*(A_2) \leq P^*(A_1 \cup A_2) + P^*(A_1 \cap A_2). \]

*Proof.* Using Definition 6, the proof is straightforward.

Note that the example of an upper-lower functional \((P_*, P^*)\) given above that is not an envelope does have properties (i) and (ii) of Lemma 8 when \(A_1 \cap A_2 = \phi\).

**Lemma 9.** If the upper-lower functional \((P_*, P^*)\) is a capacity of order two, then for each subfamily \(\{A_i\}\) of \(\mathcal{B}\) we have:

(i) \[ \sum P^*(A_i) \geq P^*\left( \bigcup_i A_i \right) + P^*\left( \bigcup_{i \neq j} (A_i \cap A_j) \right) + \cdots + P^*\left( \bigcap_i A_i \right), \]
(ii) \[ \sum P_*(A_i) \leq P_*(\bigcup_i A_i) + P_*(\bigcup_{i \neq j} (A_i \cap A_j)) + \cdots + P_*(\bigcap_i A_i). \]

*Proof.* For (i) we proceed by induction on \(n\). For the case \(n = 2\) it follows from the definition of capacity of order two that:

\[ P^*(A_1) + P^*(A_2) \geq P^*(A_1 \cup A_2) + P^*(A_1 \cap A_2). \]

The result for \(n + 1\) follows from the case \(k \leq n\) by repeated applications of the case \(n = 2\) to the following inequality:

\[ \sum_{i=1}^{n+1} P^*(A_i) \geq P^*\left( \bigcup_{i=1}^n A_i \right) + P^*\left( \bigcup_{i \neq j}^n (A_i \cap A_j) \right) + \cdots + P^*\left( \bigcap_{i=1}^n A_i \right) + P^*(A_{n+1}). \]

Similarly for (ii).

**Lemma 10.** If the upper-lower functional \((P_*, P^*)\) is a capacity of order two, then for each subfamily \(\{A_i\}\) of \(\mathcal{B}\) we have:
(i) \[ P_\ast(A_j) + \sum_{i \neq j} P_\ast(A_i) \geq P_\ast \left( \bigcup_i A_i \right) + P_\ast \left( \bigcup_{i < j} (A_i \cap A_j) \right) + \cdots + P_\ast \left( \bigcap_i A_i \right) \]

(ii) \[ P_\ast(A_j) + \sum_{i \neq j} P_\ast(A_i) \leq P_\ast \left( \bigcup_i A_i \right) + P_\ast \left( \bigcup_{i \neq j} (A_i \cap A_j) \right) + \cdots + P_\ast \left( \bigcap_i A_i \right) \]

Proof. We will prove (i); (ii) follows similarly. From Lemma 9 we can write:

\[ P_\ast(A_{n+1}) + \sum_{i=1}^n P_\ast(A_i) \geq P_\ast \left( \bigcup_{i=1}^n A_i \right) + P_\ast \left( \bigcup_{i \neq j} (A_i \cap A_j) \right) + \cdots + P_\ast \left( \bigcap_{i=1}^n A_i \right) + P_\ast(A_{n+1}). \]

The result follows by repeated applications of property (i) in Lemma 8.

LEMMA 11. For each \( f \in G(\mathcal{B}) \) we have

(i) \[ F_\ast(f) = \mu_\ast(f), \]
(ii) \[ F^\ast(f) = \mu^\ast(f), \]

iff \((P_\ast, P^\ast)\) is a capacity of order two.

Proof. If \( \sum A_i^c \) is any representation of \( f \), it follows from Lemma 9 that

\[ \sum P_\ast(A_i) \leq P_\ast \left( \bigcup_i A_i \right) + P_\ast \left( \bigcup_{i \neq j} (A_i \cap A_j) \right) + \cdots + P_\ast \left( \bigcap_i A_i \right) = \mu_\ast(f). \]

Then

\[ F_\ast(f) = \text{SUP} \left\{ \sum P_\ast(A_i) : \sum A_i^c = f \right\} \leq \mu_\ast(f). \]

Clearly \( F_\ast(f) \geq \mu_\ast(f) \). Similarly, \( F^\ast(f) = \mu^\ast(f) \). We have thus proved the following:

PROPOSITION 1. The functionals \( \mu_\ast \) and \( \mu^\ast \) will be superadditive and subadditive respectively iff the upper-lower functional \((P^\ast, P_\ast)\) is a capacity of order two.

We now prove the main theorem of this section.
THEOREM 3. If the upper-lower functional \((P_*, P^*)\) is a capacity of order two, then it is an envelope.

Proof. Let \(\sum A_i^c\) and \(\sum B_i^c\) be two representations for \(f \in G(\mathcal{B})\) with \(A = A_i\) for \(i \leq m\). Then from Lemma 10 and the fact that the spectral representation of \(f = mA^c\) is \(mA^c\), we have:

\[
mP_*(A) + \sum_{i > m} P^*(A_i) \geq P_*(\bigcup_i A_i) + P_*(\bigcup_{i \neq j} (A_i \cap A_j)) + \cdots + P_*(\bigcap_i A_i)
\]

\[
= P_*(\bigcup_i B_i) + P_*(\bigcup_{i \neq j} (B_i \cap B_j)) + \cdots + P_*(\bigcap_i B_i)
\]

\[
= \sum_i P_*(B_i), \text{ by Lemma 9.}
\]

Similarly

\[
mP^*(A) + \sum_{i > m} P_*(A_i) \leq \sum_i P^*(B_i).
\]

DEFINITION 7. Let \(\mathcal{A}\) be a Boolean algebra on \(\Omega\), with \(\mathcal{B}\) a subalgebra of \(\mathcal{A}\). The inner-outer probability functional \((P_*, P^*)\) on the probability space \((\Omega, \mathcal{B}, P)\) is the pair of set functions on \(\mathcal{A}\):

\[
P^*(A) = \inf\{P(B); A \subseteq B \in \mathcal{B}\},
\]

\[
P_*(A) = \sup\{P(B); A \supseteq B \in \mathcal{B}\},
\]

where \(A\) is in \(\mathcal{A}\).

It is clear from this definition that when \(A\) is in \(\mathcal{B}\), \(P_*(A) = P^*(A) = P(A)\).

LEMMA 12. An inner-outer probability functional \((P_*, P^*)\) is a capacity of order two.

Proof. Clearly, \(P^*(A) = 1 - P_*(\overline{A})\). We must show that

\[
P_*(A_1) + P_*(A_2) \leq P_*(A_1 \cup A_2) + P_*(A_1 \cap A_2).
\]

Only the infinite case requires proof. Fix \(\epsilon > 0\) and choose \(B_1, B_2\) in \(\mathcal{B}\).
such that $B_i \subseteq A_i$ and

$$P_*(A_i) - \frac{\epsilon}{2} \leq P(B_i) \quad \text{for } i = 1, 2.$$

Then

$$P_*(A_1) + P_*(A_2) - \epsilon \leq P(B_1) + P(B_2)$$
$$= P(B_1 \cup B_2) + P(B_1 \cap B_2)$$
$$\leq P_*(A_1 \cup A_2) + P_*(A_1 \cap A_2).$$

PROPOSITION 2 (Kelly). Let $(\Omega, \mathcal{B}, P)$ be a finitely additive probability space. Let $\mathcal{A}$ be a Boolean algebra of subsets of $\Omega$ containing $\mathcal{B}$. Then $P$ can be extended to $\mathcal{A}$.

Proof. It follows from Lemma 12 and Theorem 3.

This extension theorem for finitely additive probability measures shows an advantage of de Finetti's approach to probability theory, because the extension can always be made, which is not always possible for countably additive measures. This theorem is well known. We included it here as an application of Theorem 3, which leads to a very simple proof.

4. THE FOURTH THEOREM

In the first part of this section, we follow the sequence of concepts developed in Suppes and Zanotti (1977).

Let $X$ and $Y$ be two nonempty sets. Then the set $\mathcal{R}(X, Y)$ is the set of all (binary) relations $R \subseteq X \times Y$. We shall also occasionally refer to such a relation $R$ as a multivalued mapping from $X$ into $Y$, which is the terminology used by Dempster (1967). It is obvious that $\mathcal{R}(X, Y)$ is a Boolean algebra under the operations of intersection, union and complementation. The domain of a relation $R$ is defined as

(1) $\mathcal{D}(R) = \{x : (\exists y)(xRy)\}$,

and the notion of range is defined similarly.

(2) $\mathcal{R}(R) = \{y : (\exists x)(xRy)\}$.

The domain function $\mathcal{D}$ may also be thought of as a mapping from $\mathcal{R}(X, Y)$ to the power set, $\mathcal{P}(X)$, of $X$, and the range function as a mapping from $\mathcal{R}(X, Y)$ to $\mathcal{P}(Y)$.  


Because of the symmetry in the domain and range mappings, we list explicitly only the properties of the domain mapping:

(3) \( \mathcal{D}(\emptyset) = \emptyset \), where \( \emptyset \) is the empty set, which is also the empty relation,
(4) \( \mathcal{D}(U) = X \), where \( U = X \times Y \) is the universal relation,
(5) \( \mathcal{D}(R_1 \cup R_2) = \mathcal{D}(R_1) \cup \mathcal{D}(R_2) \), for \( R_1, R_2 \in \mathbb{R}(X, Y) \),
(6) \( \mathcal{D}(R_1 \cap R_2) \subseteq \mathcal{D}(R_1) \cap \mathcal{D}(R_2) \),
(7) \( \mathcal{D}(R_1) \sim \mathcal{D}(R_2) \subseteq \mathcal{D}(R_1 \sim R_2) \), where \( \sim \) is set difference.

For several purposes it is convenient to have a restricted form of complementation: For \( R \in \mathbb{R}(X, Y) \) the complement \( \neg R \) is with respect to \( X \times Y \), i.e.,

(8) \( \neg R = (X \times Y) \sim R \),

the complement of \( A \subseteq X \) is \( X \sim A \), and the complement of \( B \subseteq Y \) is \( Y \sim B \). Thus \( \neg \mathcal{D}(R) = X \sim \mathcal{D}(R) \). The point to note is that unrestricted complementation of sets is of no interest in the present context, i.e., it is of no interest to have the complementation of \( R \in \mathbb{R}(X, Y) \) and \( \mathcal{D}(R) \) relative to the same universe.

We next turn to some familiar operations on relations, or on relations and sets. The converse or inverse of a relation is defined as

(9) \( \tilde{R} = \{(y, x) : xRy\} \).

This notion is, of course, the relational generalization of function inverse. Familiar properties for \( R, R_1 \) and \( R_2 \) in \( \mathbb{R}(X, Y) \) are these:

(10) \( \tilde{R} = R \),
(11) \( R_1 \cap R_2 = \tilde{R}_1 \cap \tilde{R}_2 \),
(12) \( R_1 \cup R_2 = \tilde{R}_1 \cup \tilde{R}_2 \),
(13) \( R_1 \sim R_2 = \tilde{R}_1 \sim \tilde{R}_2 \).

The notion of a relation's domain being restricted to a given set is defined as

(14) \( R \upharpoonright A = R \cap (A \times \mathcal{R}(R)) \).

We next turn to two concepts that are especially important for subsequent developments. The first, \( R^n A \), is ordinarily called the image of \( A \) under the relation \( R \), but, for reasons that will soon be made clear, we shall call it the upper image of \( A \) under \( R \). The definition is simple in terms of restriction and range.
but more suggestive is the equivalence
\[ y \in R''A \iff (\exists x)(xRy \text{ and } x \in A). \]

Now let us define, for immediate comparison, the less standard notation of lower image, introduced in analogy to the relation between upper and lower probabilities.

\[ P_*(A) = 1 - P^*(\neg A). \]

Thus, we have

\[ R_\nu A = \neg(R''\neg A). \]

The 'outside' complementation of (18) is with respect to \( Y \), and the 'inside' one with respect to \( X \). In order to have, again in analogy to the case of upper and lower probabilities, the inequality corresponding to

\[ P_*(A) \leq P^*(A), \]

we need for the range of \( R \) to be \( Y \), and in the case of the inverse image, the range of \( \tilde{R} \) to be \( X \).

(20) If \( R(R) = Y \) then \( R_\nu A \subseteq R''A \).

(21) If \( R(\tilde{R}) = X \) then \( \tilde{R}_\nu B \subseteq \tilde{R}''B \).

This restriction is a natural one, for it corresponds to a multivalued mapping having all of \( X \) as its domain, a point that is expanded on below.

The familiar superadditive and subadditive properties of upper and lower probabilities are expressed in the inequalities: For \( A \cap B = \emptyset \),

\[ P_*(A) + P_*(B) \leq P_*(A \cup B) \leq P^*(A \cup B) \leq P_*(A) + P^*(B). \]

As the relational analogue we have:

(23) \( (R_\nu A) \cup (R_\nu B) \subseteq R_\nu (A \cup B) \),

(24) \( R''(A \cup B) = (R''A) \cup (R''B) \),

but (23) and (24) are not restricted to \( A \cap B = \emptyset \). Some other properties of the upper and lower images of a set are the following:

(25) \( R_\nu (A \cap B) = (R_\nu A) \cap (R_\nu B) \),
(26) \( R'(A \cap B) \subseteq (R''A) \cap (R''B), \)
(27) If \( A \subseteq B \) then \( R''A \subseteq R''B, \)
(28) If \( A \subseteq B \) then \( R''A \subseteq R''B, \)
(29) \( R''\phi = R'' \phi = \phi, \)
(30) \( R''X = R'X = R(R). \)

Note that in (30) \( R(R) \) plays the role of the universe in the image sample space. On the basis of (25) the lower image is a homomorphism with respect to the intersection of sets, and on the basis of (24) the upper image is such a mapping with respect to the union of sets.

We now turn to relations between Boolean algebras on \( X \) and \( Y \). Given \( R \in R(X, Y) \) and a Boolean algebra \( B \) of subsets of \( Y \), the class

\[
(31) \quad C_* = \{A : A \subseteq X \land (\exists B)(B \in B \land R'B = A)\}
\]

is a \( \tau \)-system of subsets of \( X \), i.e., it is closed under intersection, and the class

\[
(32) \quad C^* = \{A : A \subseteq X \land (\exists B)(B \in B \land R''B = A)\}
\]

is a family of subsets of \( X \) closed under union. The classes \( C_* \) and \( C^* \) are said to be induced from \( B \) by \( R \). If \( R \) is a function from \( X \) to \( Y \), then \( C_* \) and \( C^* \) are Boolean algebras and \( C_* = C^* \).

It is clear that \( C_* \) and \( C^* \) each generate Boolean algebras on \( X \), by adding closure under complementation. We have the following:

**Lemma 13.** Let \( B(C_*) \) and \( B(C^*) \) be the Boolean algebras on \( X \) generated by \( C_* \) and \( C^* \), respectively. Then

\[
(33) \quad B(C_*) = B(C^*).
\]

We next introduce the concept of measurable relation, which is a natural generalization of the standard concept of measurable function. Recall first that a measurable space \( (X, B) \) consists of a nonempty set \( X \) and a Boolean algebra of subsets of \( X \). Given two measurable spaces \( (X, B_1) \) and \( (Y, B_2) \), a relation \( R \in R(X, Y) \) is said to be \( (B_1, B_2) \)-measurable if \( R''B_2 \) and \( R''B_2 \) are contained in \( B_1 \). Here extension of the upper and lower image notation to families of sets is obvious; e.g.,

\[
(34) \quad R''B_2 = \{A : A \subseteq X \land (\exists B)(B \in B_2 \land R'B = A)\}.
\]

We then have the following lemma.
LEMMA 14. In order that $R \in \mathcal{R}(X, Y)$ be $(\mathcal{B}_1, \mathcal{B}_2)$-measurable it suffices that either $\tilde{R}_n \mathcal{B}_2 \subseteq \mathcal{B}_1$ or $\tilde{R}'' \mathcal{B}_2 \subseteq \mathcal{B}_1$.

Given a measurable space $(Y, \mathcal{B}_2)$, a probability space $\mathcal{X} = (X, \mathcal{B}_1, P)$ and a $(\mathcal{B}_1, \mathcal{B}_2)$-measurable relation $R \in \mathcal{R}(X, Y)$, we define for $A \in \mathcal{B}_2$

$$
(35) \quad P^*(A) = P(\tilde{R}_n A) \\
P^*(A) = P(\tilde{R}'' A).
$$

We call the pair $(P^*, P^*)$ a Dempsterian functional (generated by $\mathcal{X}$ and $R$) after Dempster (1967).

An upper-lower functional $(P^*, P^*)$ on a measurable space $(Y, \mathcal{B})$ is said to be a capacity of order $n$ iff the following conditions are satisfied for all $A, A_1, \ldots, A_n$ in $\mathcal{B}$:

1. $P^*(A) - \sum_i P^*(A \cap A_i) + \sum_{i<j} P^*(A \cap A_i \cap A_j) + \cdots + (-1)^nP^*(A \cap A_1 \cdots \cap A_n) \geq 0.$
2. $P^*(A) = 1 - P^*(\neg A).$

Obviously, if $(P^*, P^*)$ is a capacity of order $n$, then it is a capacity of order $m \leq n$. In addition, we say that $(P^*, P^*)$ is a capacity of infinite order if it is a capacity of order $n$ for all $n \geq 1$. As mentioned earlier, the concept of capacity is thoroughly studied by Choquet (1955). We have the following proposition relating Dempsterian functionals and capacities of infinite order.

PROPOSITION 3. Given a measurable space $(Y, \mathcal{B}_2)$, a probability space $\mathcal{X} = (X, \mathcal{B}_1, P)$, and a $(\mathcal{B}_1, \mathcal{B}_2)$-measurable relation $R \in \mathcal{R}(X, Y)$, then the Dempsterian functional $(P^*, P^*)$ on $(Y, \mathcal{B}_2)$ generated by $\mathcal{X}$ and $R$ is a capacity of infinite order.

Proof. We will show that $(P^*, P^*)$ is superadditive of arbitrary order. Given $A_1, \ldots, A_n$ in $\mathcal{B}_2$ we have $\tilde{R}_n A_1, \ldots, \tilde{R}_n A_n$ in $\mathcal{B}_1$. Then

$$
P\left(\bigcup_i \tilde{R}_n A_i\right) - \sum_i P(\tilde{R}_n A_i) + \sum_{i<j} P(\tilde{R}_n A_i \cap \tilde{R}_n A_j) + \cdots + (-1)^nP\left(\bigcap_i \tilde{R}_n A_i\right) = 0.
$$
Recalling that \( \bigcup \tilde{N} A_i \subseteq \tilde{N}(\bigcup A_i) \) and \( \tilde{N}(\bigcap A_i) = \bigcap \tilde{N} A_i \) we have (i).

For (ii)

\[
P^*(A) = P(\tilde{N} A) = P(\tilde{N} \tilde{A}) \\
= 1 - P(\tilde{N} \tilde{A}) \\
= 1 - P^*(\tilde{A}).
\]

The converse of this proposition is true, but for the sake of simplicity we will prove it later, but only for the finite case.

**DEFINITION 8.** Let \((\Omega, B)\) and \((\mathcal{B}, \mathcal{P}(\mathcal{B}))\) be given where \(\Omega\) is a nonempty set, \(\mathcal{B}\) a Boolean algebra of subsets of \(\Omega\), and \(\mathcal{P}(\mathcal{B})\) the power set of \(\mathcal{B}\). Define two set-valued set functions

\[
B \xrightarrow{(*)^*} \mathcal{P}(\mathcal{B}) \quad \text{as follows:}
\]

For each \(A\) in \(\mathcal{B}\),

\[
(A)^* = \{B : B \in \mathcal{B}, B \subseteq A\} \\
(A)^* = \{B : B \in \mathcal{B}, B \cap A \neq \emptyset\}.
\]

These mappings are the *lower mapping* and *upper mapping* of \(\mathcal{B}\) into \(\mathcal{P}(\mathcal{B})\), respectively.

**LEMMA 15.** The lower mapping \((\_\_)^*\) satisfies the following properties:

1. It is injective,
2. \(A \subseteq B\) iff \((A)^* \subseteq (B)^*\), i.e., it is monotonic,
3. \((\emptyset)^* = \{\emptyset\},\)
4. \((\Omega)^* = \mathcal{B},\)
5. \((A)^* \cup (B)^* \subseteq (A \cup B)^*,\)
6. \((A)^* \cap (B)^* = (A \cap B)^*.\)

Thus \((\_\_)^*\) is a homomorphism for intersection.

The proofs of these results are immediate.

**LEMMA 16.** The upper mapping \((\_\_)^*\) satisfies the following properties:

1. It is injective.
(2) \( A \subseteq B \) iff \((A)^* \subseteq (B)^*\), i.e., it is monotonic,
(3) \((\phi)^* = \phi\),
(4) \((\Omega)^* = \mathcal{B} - \{\phi\}\),
(5) \((A)^* \cup (B)^* = (A \cup B)^*\),
(6) \((A)^* \cap (B)^* \supseteq (A \cap B)^*\).

Thus \((\cdot)^*\) is a homomorphism for union.

The proofs of these results are immediate.

For notational convenience we will write \( A_* \) and \( A^* \) for \((A)_*\) and \((A)^*\), respectively when no confusion will arise.

**Lemma 17.** We have

(i) \( (A^*) = (\bar{A})_* \) and \( (\bar{A})^* = (A_*) \),
(ii) \( A^* \cap B^* = \phi \) iff \( A = \phi \) or \( B = \phi \),
(iii) \( A_* \cup B_* \subseteq (A \cup B)_*\),
(iv) \( A_* \cup B_* \subseteq (A \cup B)^*\),
(v) \( (A \cup B)_* \subseteq A_* \cup B_*\),
(vi) \( A_* \cap B_* = \phi \) iff \( A \cap B = \phi \),
(vii) \( A_* \cap B_* = \phi \) iff \( A \cap B = \phi \).

In particular the following chain of inclusions is true for all \( A, B \) in \( \mathcal{B} \):

\[ A_* \cup B_* \subseteq (A \cup B)_* \subseteq A_* \cup B_* \subseteq (A \cup B)^* = A^* \cup B^*. \]

The proof of the lemma is lengthy but straightforward.

The following lemma gives a characterization of the set of images of the lower mappings.

**Lemma 18.** Let \( \mathcal{C} = \{A_* : A \in \mathcal{B}\} \). Then \( \mathcal{C} \) is a Dynkin’s \( \pi \)-system of subsets of \( \mathcal{B} \).

**Proof.** If \( A_* \), \( B_* \in \mathcal{C} \) we have \( A_* \cap B_* = (A \cap B)_* \in \mathcal{C} \) since \( A \cap B \in \mathcal{B} \).

**Lemma 19.** Assume \( \mathcal{B} \) finite and let \( \mathcal{D} = \{A_*, A^* : A \in \mathcal{B}\} \). Then the Boolean algebra generated by \( \mathcal{D} \) is \( \mathcal{P}(\mathcal{B}) \).

**Proof.** For each \( A \) in \( \mathcal{B} \) there is a family \( \{A_i\} \) of atoms of \( \mathcal{B} \) such that we can uniquely write

\[ A = \bigcup_{i=1}^{n} A_i. \]
Furthermore we can write:

\[ \{A\} = \left( \bigcap_{i=1}^{n} A_i^* \right) \cap \left( \bigcup_{i=1}^{n} A_i \right)^*. \]

In fact

\[ \bigcup_{i=1}^{n} A_i \in A_i^* \]

for each \( i \leq n \) and

\[ \bigcup_{i \neq j} A_i \notin A_j^* \]

for each \( j \leq n \).

**DEFINITION 9.** The triple \((\Omega, \mathcal{B}, (P_*, P^*))\) consisting of a set \(\Omega\), a Boolean algebra \(\mathcal{B}\) of subsets of \(\Omega\) and a Dempsterian functional \((P_*, P^*)\) on \(\mathcal{B}\), is a **Dempsterian space**.

We will say that \((\Omega, \mathcal{B}, (P_*, P^*))\) is a **finite** Dempsterian space if \(\Omega\) is a finite set.

**LEMMA 20.** The triple \((\Omega, \mathcal{B}, (P_*, P^*))\) is a finite Dempsterian space if and only if there is a function \(f: \mathcal{B} \to [0, 1]\) satisfying the following properties:

(i) \(0 \leq f(A) \leq 1\) for every \(A \in \mathcal{B}\), and \(f(\phi) = 0\),

(ii) \(\sum_{A \in \mathcal{B}} f(A) = 1\).

*Proof.* Given \(A \in \mathcal{B}\) we have \(A = \bigcup A_i\) with \(A_i\)'s atoms of \(\mathcal{B}\). Let \(B_i = \bigcup_{j \neq i} A_j\). From superadditivity we can define \(f\) explicitly:

\[ f(A) = P_*(A) - \sum_i P_*(B_i) + \sum_{i < j} P_*(B_i \cap B_j) \]

\[ + \cdots + (-1)^n P_*\left( \bigcap_{i} B_i \right) \geq 0. \]

It is simple to show that

\[ \sum_{A \in \mathcal{B}} f(A) = P_*(\Omega) = 1. \]

Conversely, given a function \(f: \mathcal{B} \to [0, 1]\) we can define for all \(A\) in \(\mathcal{B}\)
\[ P_*(A) = \sum_{B \subseteq A, B \in \mathcal{B}} f(B) \]

and

\[ P^*(A) = \sum_{B \cap A \neq \emptyset, B \in \mathcal{B}} f(B) \]

and show that \((P_*, P^*)\) is a Dempsterian functional on \(\mathcal{B}\). We will not do this here, since the proof of the converse follows from the following lemma:

**Lemma 21.** Given a finite measurable space \((\Omega, \mathcal{B})\) and a probability space \((\mathcal{B}, \mathcal{P}(\mathcal{B}), P)\), define for each \(A\) in \(\mathcal{B}\)

\[
\begin{align*}
P_*(A) &= P(A_*), \\
P^*(A) &= P(A^*).
\end{align*}
\]

The pair \((P_*, P^*)\) is a Dempsterian functional on \(\mathcal{B}\).

**Proof.**

\[
P^*(A) = P(A^*) = P(\overline{A}^*) \text{ by Lemma 17},
\]

\[
= 1 - P((A)_*) = 1 - P_*(A).
\]

To \(A_1, A_2, \ldots, A_n\) in \(\mathcal{B}\) there correspond their images \((A_1)_*, (A_2)_*, \ldots, (A_n)_*\) in \(\mathcal{P}(\mathcal{B})\). We have:

\[
P\left(\bigcup_i (A_i)_*\right) - \sum_i P(A_i)_* + \sum_{i < j} P((A_i)_* \cap (A_j)_*) + \cdots + (-1)^nP\left(\bigcap_i (A_i)_*\right) = 0.
\]

By Lemma 15(6) we have:

\[
P\left(\bigcup_i (A_i)_*\right) - \sum_i P(A_i)_* + \sum_{i < j} P(A_i \cap A_j)_* + \cdots + (-1)^nP\left(\bigcap_i A_i_\right) = 0.
\]

From \(\bigcup (A_i)_* \subseteq (\bigcup A_i)_*\), which is (iii) of Lemma 17, we have:

\[
P_*(\bigcup A_i) - \sum_i P_*(A_i) + \sum_{i < j} P_*(A_i \cap A_j) + \cdots + (-1)^nP_*(\bigcap_i A_i) \geq 0.
\]
Observe that any function $f: \mathcal{B} \rightarrow [0, 1]$ with $f(\emptyset) = 0$ and $\sum_{A \in \mathcal{B}} f(A) = 1$ defines a probability measure on $\mathcal{P}(\mathcal{B})$. Thus we have proved the following representation – the first part ((i)) of Theorem 4.

**THEOREM 4.** (i) There is a 1:1 correspondence between finite Dempsterian measures on $(\Omega, \mathcal{B})$ and probability measures on $(\mathcal{B}, \mathcal{P}(\mathcal{B}))$.

(ii) Given a finite Dempsterian space $(\Omega, \mathcal{B}, (P_*, P^*))$ there exists a probability space $(\Omega_1, \mathcal{B}_1, P)$ and a unique $(\mathcal{B}_1, \mathcal{B})$-measurable random relation $R(\Omega_1, \Omega)$ generating it.

*Proof.* For the proof of (ii), let

$$\Omega_1 = \mathcal{B} \quad \text{and} \quad \mathcal{B}_1 = \mathcal{P}(\mathcal{B})$$

following our earlier construction. Then, by (i) there is a unique probability measure $P$ on $\mathcal{B}_1$ with the property $P(A_*) = P_*(A)$ and $P(A^*) = P^*(A)$ for all $A$ in $\mathcal{B}$. Second, we define the random relation $R$ as follows:

$$R = \{(A, \omega): A \in \mathcal{B} \& A \neq \emptyset \& \omega \in A\}.$$

Clearly

$$\check{R}'' A = A^*$$

and

$$\check{R}'' A = A_*.$$

Moreover,

$$P^*(A) = P(\check{R}'' A),$$

and similarly for $P_*$, which completes the proof.

In one clear sense, Theorem 4 is a generalization of Kolmogorov’s representation theorem for random variables, i.e., if a random quantity in the sense of de Finetti has a distribution, then there exists a probability space and a real-valued function generating the distribution. We call the function a random variable, which is a representation of the random quantity. What we have done in Theorem 4(ii) is to generalize from random variables to random relations, and such relations express the indeterminacy characteristic of Dempsterian functionals. Of course, this generalization has a price, e.g., the theory
of upper and lower conditional probability is not entirely satisfactory (see the last section of Suppes and Zanotti 1977).

An axiomatic qualitative theory of random quantities is developed in Suppes and Zanotti (in press).

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Institute for Mathematical Studies
in the Social Sciences
Stanford University
Ventura Hall
Stanford, CA 94305
U.S.A.